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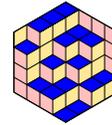
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ABSTRACT The lattice of flats \mathcal{L}_M of a matroid M is combinatorially well-behaved and, when M is realizable, admits a geometric model in the form of a “Schubert variety of hyperplane arrangement”. In contrast, the lattice of flats of a polymatroid exhibits many combinatorial pathologies and admits no similar geometric model.

We address this situation by defining the lattice \mathcal{L}_P of “combinatorial flats” of a polymatroid P . Combinatorially, \mathcal{L}_P exhibits good behavior analogous to that of \mathcal{L}_M : it is graded, determines P when P is simple, and is top-heavy. When P is realizable over a field of characteristic 0, we show that \mathcal{L}_P is modeled by “the Schubert variety of a subspace arrangement”.

Our work generalizes a number of results of Ardila–Boocher and Huh–Wang on Schubert varieties of hyperplane arrangements; however, the geometry of Schubert varieties of subspace arrangements is noticeably more complicated than that of Schubert varieties of hyperplane arrangements. Many natural questions remain open.

1. INTRODUCTION

1.1. Let M be a matroid of rank d on an N -element set E , and let \mathcal{L}_M be its poset of flats. Write \mathcal{L}_M^k for the set of flats of rank k . Beloved combinatorial properties of \mathcal{L}_M include:

- \mathcal{L}_M is a semimodular lattice (in particular, it is graded), and
- there is a unique simple matroid M^{sim} with $\mathcal{L}_{M^{\text{sim}}} \cong \mathcal{L}_M$.
- $|\mathcal{L}_M^k| \leq |\mathcal{L}_M^{k+1}|$ when $k \leq d/2n$.
- \mathcal{L}_M is **top-heavy**: $|\mathcal{L}_M^k| \leq |\mathcal{L}_M^{d-k}|$ when $k \leq d/2$.

Top-heaviness was long conjectured [10, 11], but proved only recently in [5]. The approach of [5] is inspired by [19], who proved top-heaviness when M is the matroid associated to a linear subspace $V \subset \mathbb{K}^N$, with \mathbb{K} a field. Central to [19] is the **Schubert variety of hyperplane arrangement** Y_V , defined in [2] as the closure of $V \subset \mathbb{K}^N$ in $(\mathbb{P}^1)^N = (\mathbb{K} \cup \infty)^N$.

The Schubert variety of hyperplane arrangement Y_V may be thought of as an algebro-geometric counterpart of \mathcal{L}_M , fully capturing the combinatorics of M . To see

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KEYWORDS. polymatroid, subspace arrangement, lattice of flats, Schubert variety, vector group, equivariant compactification.

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why, first observe that the additive action of \mathbb{K} on itself extends to an action on \mathbb{P}^1 , fixing ∞ . Hence, \mathbb{K}^N acts on $(\mathbb{P}^1)^N$. The orbits of this action are of the form

$$O_S = \prod_{i \in S} \mathbb{K} \times \prod_{j \in E \setminus S} \infty,$$

for each $S \subset \{1, \dots, N\}$. From [2, 19], we learn that $Y_V \cap O_S$ is nonempty if and only if S is a flat. In particular, Y_V is a union of finitely many V -orbits. Moreover, if S and S' are flats, then

- $Y_V \cap O_S$ is a single V -orbit, isomorphic to $\text{rk}_M(S)$ -dimensional affine space,
- $\overline{Y_V \cap O_S} \supset Y_V \cap O_{S'}$ if and only if $S \supset S'$, and
- the class of $\overline{Y_V \cap O_S}$ in the Chow ring of $(\mathbb{P}^1)^n$ is

$$[\overline{Y_V \cap O_S}] = \sum_{B \text{ basis of } S} [(\mathbb{P}^1)^B],$$

where $(\mathbb{P}^1)^B = \prod_{i \in B} \mathbb{P}^1 \times \prod_{j \in E \setminus B} \infty$.

1.2. Polymatroids are to subspace arrangements as matroids are to hyperplane arrangements. If M is replaced by a rank- d polymatroid P on E , then all expected combinatorial properties fail. The poset of flats of a polymatroid is always a lattice, but may not be graded. When P is simple, the poset fails to determine P , and there is no sense in which it is top-heavy.

To remedy these pathologies, we introduce the **lattice of combinatorial flats** \mathcal{L}_P of a polymatroid. When P is a matroid, \mathcal{L}_P is the usual lattice of flats. When P is realized by a subspace arrangement V_1, \dots, V_N , the lattice \mathcal{L}_P is the poset of flats of a collection of general flags with minimal elements V_1, \dots, V_N (see Example 1.2).

For arbitrary polymatroids, \mathcal{L}_P may be constructed as follows⁽¹⁾. Let rk_P be the rank function of P , and let $\mathbf{e}_1, \dots, \mathbf{e}_N$ be the standard basis of \mathbb{Z}^N . Choose $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}_{\geq 0}^N$, such that $n_i \geq \text{rk}_P(i)$. Recall that an **independent multiset** of P is $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{Z}_{\geq 0}^N$ such that $\sum_{i \in A} b_i \leq \text{rk}_P(A)$ for all $A \subset E$. A **basis** for a multiset $\mathbf{s} = (s_1, \dots, s_N) \in \mathbb{Z}_{\geq 0}^N$ is an independent multiset $\mathbf{b} = (b_1, \dots, b_N)$, maximal among those contained in \mathbf{s} . Define the **rank** of \mathbf{s} by $\text{rk}_P(\mathbf{s}) := \sum_{i \in E} b_i$ for any basis \mathbf{b} ⁽²⁾, and call \mathbf{s} a **combinatorial flat**⁽³⁾ if $\text{rk}_P(\mathbf{s} + \mathbf{e}_i) > \text{rk}_P(\mathbf{s})$ for all i such that $s_i < n_i$.

We define \mathcal{L}_P to be the set of combinatorial flats, ordered by inclusion. We will see in Section 3.4 that \mathcal{L}_P does not depend on \mathbf{n} (although the underlying set of combinatorial flats does; we write $\mathcal{L}_{P,\mathbf{n}}$ when we wish to emphasize this).

THEOREM 1.1. *Let P be a polymatroid. The poset of combinatorial flats satisfies:*

- (1) \mathcal{L}_P is a graded semimodular lattice,
- (2) there is a unique simple polymatroid P^{sim} such that $\mathcal{L}_P \cong \mathcal{L}_{P^{\text{sim}}}$,
- (3) $|\mathcal{L}_P^k| \leq |\mathcal{L}_P^{k+1}|$ when $k \leq d/2$, and
- (4) \mathcal{L}_P is top-heavy: $|\mathcal{L}_P^k| \leq |\mathcal{L}_P^{d-k}|$ when $k \leq d/2$.

EXAMPLE 1.2. Suppose that $V_1, V_2, V_3, V_4 \subseteq V = \mathbb{C}^3$ is a linear subspace arrangement, where V_1, V_2 , and V_3 are two dimensional, $V_1 \cap V_2 \cap V_3$ is one dimensional, and V_4 is one dimensional and not contained in V_i for $i = 1, 2, 3$. This arrangement defines the polymatroid P on $E = \{1, 2, 3, 4\}$ with $\text{rk}_P(S) = 3 - \dim \bigcap_{i \in S} V_i$. Flats of P

⁽¹⁾The definitions of “rank” and “combinatorial flat” given here are different from, but equivalent to, the definitions given in Section 3.1. The equivalence is explained in Remark 3.6.

⁽²⁾Remark 3.5 explains well-definedness of rank. We write $\text{rk}_P(\cdot)$ for ranks of both sets and multisets; this will never cause confusion.

⁽³⁾This definition resembles the related notion of “flats for matricubes” [1]. See Remark 3.8 for comparison of these notions.

correspond to intersections of V_i 's. The lattice of flats is neither graded nor top-heavy (Fig. 1).

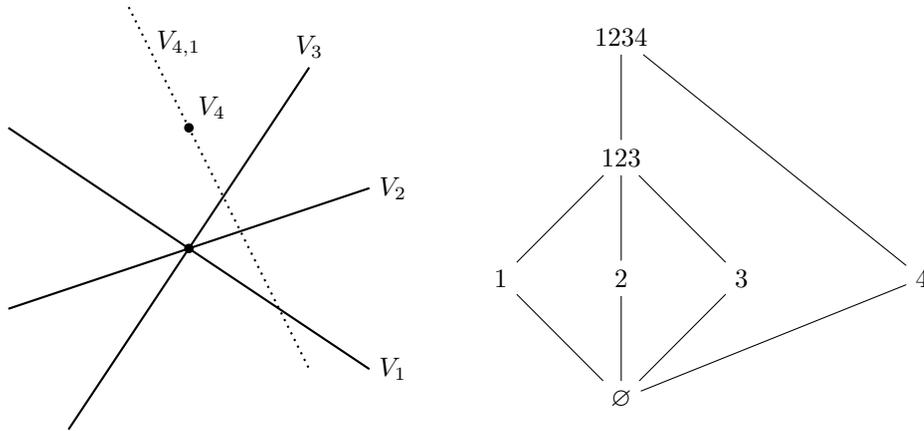


FIGURE 1. Left: the projectivization of the subspaces arrangement V_1, \dots, V_4 from Example 1.2 (solid), along with the projectivization of a general plane $V_{4,1}$ containing V_4 (dashed). Right: the lattice of flats of the polymatroid P realized by $\{V_1, V_2, V_3, V_4\}$.

On the other hand, $\mathcal{L}_P \cong \mathcal{L}_{P,\mathbf{n}}$ (with $\mathbf{n} = (1, 1, 1, 2)$) is both graded and top-heavy (Fig. 2). Elements of \mathcal{L}_P correspond to flats of the arrangement $\{V_1, V_2, V_3, V_4, V_{4,1}\}$, where $V_{4,1}$ is a general plane containing V_4 . There is a lattice homomorphism embedding the lattice of flats of P into \mathcal{L}_P . This construction is explained in full generality in Remark 3.7.

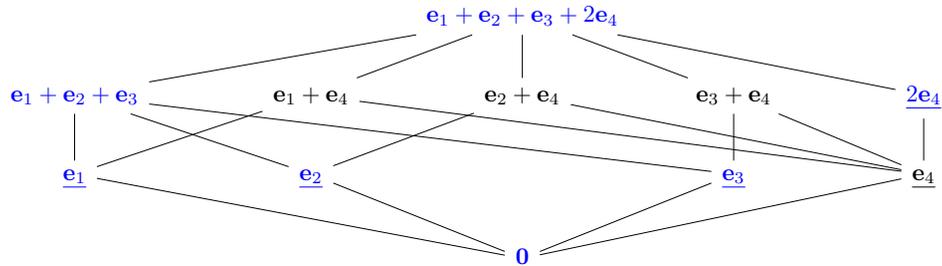


FIGURE 2. The lattice of combinatorial flats of P from Example 1.2. Join-irreducibles are underlined. Blue elements are in the image of the embedding of P 's lattice of flats into \mathcal{L}_P .

Lattices of combinatorial flats can be axiomatized in a manner that closely resembles the well-known definition of geometric lattices (Theorem 3.20). Consequently, they constitute a new cryptomorphic definition of simple polymatroids.

1.3. Unlike the usual poset of flats of P , \mathcal{L}_P can be realized as the poset of affine cells of a projective variety whenever P is realizable over a field of characteristic zero.

Work of Hassett-Tschinkel [17] (reviewed in Section 4.1) shows that there exists action of the additive group $\mathbb{K}^{\mathbf{n}} := \mathbb{K}^{n_1} \times \dots \times \mathbb{K}^{n_N}$ on $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_N}$, such that $\mathbb{P}^{\mathbf{n}}$ is partitioned into finitely many orbits $O_{\mathbf{s}}$, corresponding to multisets $\mathbf{s} \leq \mathbf{n}$.

Under the assumption that V is “polymatroid general”, a mild genericity condition described in Section 4.6 that constrains the intersection of V with certain coordinate subspaces, we define Y_V to be the V -orbit closure of the origin in \mathbb{P}^n under the Hassett-Tschinkel action of \mathbb{K}^n on \mathbb{P}^n .

THEOREM 1.3. *If $V \subset \mathbb{K}^n$ is a polymatroid general subspace realizing P , then $Y_V \cap O_{\mathbf{s}}$ is nonempty if and only if \mathbf{s} is a combinatorial flat. Moreover, if $\mathbf{s}, \mathbf{s}' \in \mathcal{L}_P \cong \mathcal{L}_{P, \mathbf{n}}$, then*

- (1) $Y_V \cap O_{\mathbf{s}}$ is a single V -orbit, isomorphic to affine space of dimension $\text{rk}_P(\mathbf{s})$,
- (2) $\overline{Y_V \cap O_{\mathbf{s}}} \supset Y_V \cap O_{\mathbf{s}'}$ if and only if $\mathbf{s} \geq \mathbf{s}'$, and
- (3) the class of $\overline{Y_V \cap O_{\mathbf{s}}}$ in the Chow ring of \mathbb{P}^n has the form

$$[\overline{Y_V \cap O_{\mathbf{s}}}] = \sum_{\mathbf{b} \text{ basis of } \mathbf{s}} c_{\mathbf{b}} [\mathbb{P}^{\mathbf{b}}],$$

with all coefficients positive.

If $\mathbb{K} = \mathbb{C}$, we also give a presentation for the singular cohomology ring of Y_V , generalizing a result of [19].

DEFINITION 1.4. *Let P be a polymatroid with ground set $E = \{1, \dots, N\}$. Given two combinatorial flats \mathbf{s} and \mathbf{s}' spanned by independent multisets \mathbf{b} and \mathbf{b}' respectively, we define their **cup product** as*

$$(\mathbf{s} \smile \mathbf{s}') := \overline{(\min\{\mathbf{b}(i) + \mathbf{b}'(i), \text{rk}_P(i)\})_{i=1}^N}.$$

REMARK 1.5. Equivalently, in terms of Definition 3.1, $\mathbf{s} \smile \mathbf{s}'$ is the combinatorial flat represented by any maximal element of $\{F \vee F'\}$ where F and F' range over flats of the multisymmetric lift \tilde{P} which represent \mathbf{s} and \mathbf{s}' respectively. Notice that the product of symmetric groups $\mathfrak{S} := \prod_{i=1}^N \mathfrak{S}_{E_i}$ acts transitively on the maximal elements of $\{F \vee F'\}$. Therefore $\mathbf{s} \smile \mathbf{s}'$ is well defined, independent of the choices of \mathbf{b} and \mathbf{b}' .

EXAMPLE 1.6. Suppose that $N = 1$, and $\text{rk}_P(\{1\}) = n$. Then the lattice \mathcal{L}_P of combinatorial flats is isomorphic to the chain of length n . Any combinatorial flat $i \in [n]$ is also independent in P , so the cup product of combinatorial flats is given as

$$i \smile j := \overline{\min(i + j, n)} = \min(i + j, n).$$

THEOREM 1.7. *If $V \subset \mathbb{C}^n$ is a polymatroid general subspace, then the singular cohomology ring of Y_V is the graded vector space*

$$H^*(Y_V; \mathbb{Q}) \cong \bigoplus_{\mathbf{s} \in \mathcal{L}_P} \mathbb{Q} \cdot y_{\mathbf{s}},$$

where $\deg y_{\mathbf{s}} = \text{rk}_P(\mathbf{s})$. There are positive constants $c_{\mathbf{b}}$ associated to bases \mathbf{b} of combinatorial flats such that multiplication is given by the formula

$$c_{\mathbf{b}} c_{\mathbf{b}'} y_{\mathbf{s}} y_{\mathbf{s}'} = \begin{cases} c_{\mathbf{b} + \mathbf{b}'} y_{\mathbf{s} \smile \mathbf{s}'}, & \text{if } \text{rk}_P(\mathbf{s} \smile \mathbf{s}') = \text{rk}_P(\mathbf{s}) + \text{rk}_P(\mathbf{s}') \\ 0, & \text{otherwise,} \end{cases}$$

where \mathbf{b} and \mathbf{b}' are bases for \mathbf{s} and \mathbf{s}' , respectively, such that $\mathbf{b} + \mathbf{b}'$ is a basis for $\mathbf{s} \smile \mathbf{s}'$.

1.4. ORGANIZATION. We review polymatroids, multisymmetric matroids, and operations on them in Section 2, before commencing study of combinatorial flats in Section 3.

We construct combinatorial flats in Section 3.1, then consider the effects of deletion and truncation in Section 3.2. These sections are the only portions of Section 3 required for Section 4. In the remainder of Section 3, we work up to a proof of Theorem 1.1 (Sections 3.3 and 3.4) and provide axioms for \mathcal{L}_P (Section 3.5)

Section 4 concerns Schubert varieties of subspace arrangements. Foundational work of Hassett and Tschinkel [17] is reviewed in Sections 4.1 and 4.2. We construct Y_V in Section 4.4, then compute the support of its Chow class in Section 4.5. In Section 4.6, we discuss “polymatroid generality”, a condition on V that guarantees Y_V consists of finitely many V -orbits. Assuming this condition, we identify strata of Y_V in Section 4.7, culminating with a proof of Theorem 1.3. In Section 4.8, we prove Theorem 1.7.

At the last, we pose a large number of questions about Schubert varieties of subspace arrangements in Section 5. Both algebraic and geometric aspects of Schubert varieties of subspace arrangements seem markedly more complicated than those of Schubert varieties of hyperplane arrangements, yet examples suggest a good theory lies in wait. Interested parties are encouraged to seek their fortune.

1.5. NOTATION. Throughout this paper, $E := \{1, \dots, N\}$. A **multiset** on E is an element of $\mathbb{Z}_{\geq 0}^E$. The **cardinality** of a multiset $\mathbf{s} = (s_1, \dots, s_N)$ is $|\mathbf{s}| := s_1 + \dots + s_N$. If $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$ is a finite set, then $S \subset \tilde{E}$ **represents** \mathbf{s} if $s_i = |S \cap E_i|$ for each $1 \leq i \leq N$. Multisets are ordered: $(s_1, \dots, s_N) \leq (s'_1, \dots, s'_N)$ if and only if $s_i \leq s'_i$ for every $1 \leq i \leq N$. Multisets will always be written in boldface.

Let \mathbb{K} be a field. Products of \mathbb{K} -vector spaces and projective spaces will be written $\mathbb{K}^{\mathbf{s}} := \mathbb{K}^{s_1} \times \dots \times \mathbb{K}^{s_N}$ and $\mathbb{P}^{\mathbf{s}} := \mathbb{P}^{s_1} \times \dots \times \mathbb{P}^{s_N}$, respectively. Like multisets, elements of $\mathbb{K}^{\mathbf{s}}$ will be written in boldface; we trust this will cause no confusion.

We write $\mathbf{e}_1, \dots, \mathbf{e}_N$ for the standard basis of \mathbb{Z}^E , and $\mathbf{0}$ for the zero vector. We will regard these as multisets or vectors as needed.

Braces for operations involving one-element sets are often omitted, e.g. we write $S \setminus j$ for $S \setminus \{j\}$ and π_j for $\pi_{\{j\}}$. The symmetric group on a set S is denoted \mathfrak{S}_S .

If $(A_i)_{i \in S}$ is a family of sets indexed by a set S , and $T \subset S$, then $\pi_T : \prod_{i \in S} A_i \rightarrow \prod_{i \in T} A_i$ is the projection. We will use the same symbol π_T for the projection out of any product indexed by a set containing T . For example, we will write $\pi_{E \setminus N}$ both for the projection $\mathbb{Z}_{\geq 0}^N \rightarrow \mathbb{Z}_{\geq 0}^{N-1}$ that forgets the last coordinate of a multiset, and for the projection $\mathbb{K}^{\mathbf{s}} \rightarrow \mathbb{K}^{\pi_{E \setminus N}(\mathbf{s})}$ that forgets the last coordinate of a vector in $\mathbb{K}^{\mathbf{s}}$. Once more, we expect this conventions will cause no confusion.

2. POLYMATROIDS

We review polymatroids, the nearly-equivalent notion of multisymmetric matroids, and some operations on them.

2.1. A **polymatroid** P on a finite set $E = \{1, \dots, N\}$ is the data of a **rank** function $\text{rk} : 2^E \rightarrow \mathbb{Z}_{\geq 0}$ that is

normalized: $\text{rk}(\emptyset) = 0$,

increasing: if $A \subset B$, then $\text{rk}(A) \leq \text{rk}(B)$, and

submodular: if $A, B \subset E$, then $\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B)$.

The **rank** of P is $\text{rk}(P) := \text{rk}(E)$, and the **corank** of a set is $\text{crk}(A) := \text{rk}(P) - \text{rk}(A)$. If $\text{rk}(A) \leq |A|$ for every $A \subset E$, then P is a **matroid**. A **flat** is a subset $F \subset E$ such that $\text{rk}(F \cup i) = \text{rk}(F)$ for all $i \in E \setminus F$. The intersection of two flats is a flat.

The **closure** of $A \subset E$ is $\bar{A} := \bigcap_{F \supset A} F$, where F runs over all flats of P containing A . The closure of A is the smallest flat that contains A , so $\text{rk}(A) = \text{rk}(\bar{A})$. Call P **loopless** if each element of E has positive rank, and **simple** if each one-element subset of E is a flat of positive rank.

A **basis** of P is a multiset $\mathbf{b} \in \mathbb{Z}_{\geq 0}^E$ satisfying $|\mathbf{b}| = \text{rk}(P)$ and $|\pi_A(\mathbf{b})| \leq \text{rk}(A)$ for all $A \subsetneq E$. A polymatroid is determined by its bases because

$$\text{rk}(A) = \max\{|\pi_A(\mathbf{b})| : \mathbf{b} \text{ is a basis of } P\}.$$

A multiset \mathbf{s} is **independent** if there is a basis \mathbf{b} such that $\mathbf{s} \leq \mathbf{b}$, equivalently, if $|\pi_A(\mathbf{s})| \leq \text{rk}(A)$ for all $A \subset E$.

When necessary, we will use subscripts to distinguish the rank functions of different polymatroids, e.g. rk_P is the rank function of P .

EXAMPLE 2.1 (Free polymatroids). Let $E = \{1, \dots, N\}$ and let $\mathbf{s} \in \mathbb{Z}_{\geq 0}^E$ be a multiset. The **free polymatroid** $B_{\mathbf{s}}$ has rank function $\text{rk}_{B_{\mathbf{s}}}(A) = \sum_{i \in A} s_i$.

EXAMPLE 2.2 (Realizable polymatroids). Let W_1, \dots, W_N be vector spaces. A linear subspace $V \subset \prod_{i=1}^N W_i$ defines a polymatroid P on $E = \{1, \dots, N\}$ with rank function

$$\text{rk}(A) := \dim \pi_A(V) = \text{codim}_V V \cap \ker(\pi_A).$$

In this case, we say V **realizes** P . The flats F of P are in inclusion-reversing bijection with subspaces $V \cap \ker(\pi_F)$. The closure of $A \subset E$ is the largest $A' \subset E$ such that $V \cap \ker(\pi_A) = V \cap \ker(\pi_{A'})$.

Likewise, a subspace arrangement V_1, \dots, V_N in a vector space V **realizes** P if $\text{rk}(A) = \text{codim}_V \bigcap_{i \in A} V_i$. These two notions of realizability are equivalent. Given a subspace $V \subset \prod_i W_i$, take $V_i = \ker \pi_i$. Given a subspace arrangement, choose vector spaces W_1, \dots, W_N and a map $f : V \rightarrow \prod_i W_i$ with $\ker(\pi_i \circ f) = V_i$. The image of f is a subspace realizing P .

EXAMPLE 2.3 (Lifts). If M is a matroid on a finite set $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$, then we may define a polymatroid P on E by $\text{rk}_P(S) := \text{rk}_M(\cup_{i \in S} E_i)$. In this case, we say that M **lifts** P .

2.2. Let $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$ be a finite set. A **multisymmetric matroid** is a matroid M on \tilde{E} whose rank function is invariant under the natural action of $\mathfrak{S} := \prod_{i=1}^N \mathfrak{S}_{E_i}$. If F is a flat of M and $\sigma \in \mathfrak{S}$, then $\sigma \cdot F$ is also a flat of M . The **geometric part** of a subset $A \subset \tilde{E}$ is $A^{\text{geo}} := \bigcap_{\sigma \in \mathfrak{S}} \sigma(A)$, and A is **geometric** if $A = A^{\text{geo}}$. If F is a flat, then F^{geo} is also a flat. Two key properties of multisymmetric matroids follow.

LEMMA 2.4 ([9, Lemma 2.8]). *Let M be a multisymmetric matroid on $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$. The following equivalent statements all hold:*

- If F is a flat of M , then $\text{rk}(F) = \text{rk}(F^{\text{geo}}) + |F \setminus F^{\text{geo}}|$.
- If $A \subset \tilde{E}$, then for each $1 \leq i \leq N$, either $\bar{A} \cap E_i = A \cap E_i$ or $\bar{A} \supset E_i$.
- If F is a flat of M , then $F \setminus e$ is a flat of rank $\text{rk}(F) - 1$ for any $e \in F \setminus F^{\text{geo}}$.

LEMMA 2.5 ([9, Lemma 2.9]). *Two multisymmetric matroids on $E_1 \sqcup \dots \sqcup E_N$ are isomorphic if and only if their geometric sets have the same ranks.*

2.3. Multisymmetric matroids and polymatroids are related by a much-rediscovered [18, 23, 21, 22, 4, 9] recipe. A **cage** for a polymatroid P on E is $\mathbf{n} \in \mathbb{Z}_{\geq 0}^E$ such that $\text{rk}(i) \leq n_i$ for all $i \in E$. The pair (P, \mathbf{n}) is a **caged polymatroid**⁽⁴⁾.

THEOREM/DEFINITION 2.1 ([9, Theorem 2.10⁽⁵⁾]). *Fix a multiset $\mathbf{n} = (n_1, \dots, n_N)$, and let $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$ be a set with $|E_i| = n_i$. There is a bijection*

$$\begin{aligned} \{\text{caged polymatroids } (P, \mathbf{n})\} &\xrightarrow{\sim} \{\text{multisymmetric matroids on } \tilde{E}\} \\ (P, \mathbf{n}) &\longmapsto \tilde{P}, \end{aligned}$$

where \tilde{P} is the **multisymmetric lift** of P , defined by

$$\text{rk}_{\tilde{P}}(A) := \min\{\text{rk}_P(B) + |A \setminus \cup_{i \in B} E_i| : B \subset E\}, \quad A \subset \tilde{E}.$$

⁽⁴⁾This terminology originates in [14].

⁽⁵⁾In [9], this is stated in the case when $\text{rk}_P(i) = |E_i|$, but the proof goes through verbatim in the present slightly more general setting.

REMARK 2.6. The matroid lift satisfies $\text{rk}_P(S) = \text{rk}_{\tilde{P}}(\cup_{i \in S} E_i)$. By applying this fact and Lemma 2.4, we learn that the closure in \tilde{P} of $\cup_{i \in S} E_i$ is $\cup_{i \in \bar{S}} E_i$, where \bar{S} is the closure of S in P .

REMARK 2.7. By Lemma 2.5, \tilde{P} is the unique multisymmetric matroid on \tilde{E} that lifts P .

It is also helpful to understand \tilde{P} in terms of bases.

PROPOSITION 2.8. *Let (P, \mathbf{n}) be a caged polymatroid with multisymmetric lift \tilde{P} . The bases of \tilde{P} are the multisets $\mathbf{b} \in \{0, 1\}^{\tilde{E}}$ with $|\mathbf{b}| = \text{rk}(P)$ such that for all $A \subset E$,*

$$|\pi_{\cup_{i \in A} E_i}(\mathbf{b})| \leq \text{rk}_P(A).$$

Equivalently, $B \subset \tilde{E}$ is a basis of \tilde{P} if and only if $(|B \cap E_1|, \dots, |B \cap E_N|)$ is a basis of P .

Proof. The equivalence of the two characterizations is apparent from the definitions. We prove the characterization by inequalities. Any basis must satisfy the proposed inequalities because $\text{rk}_P(A) = \text{rk}_{\tilde{P}}(\cup_{i \in A} E_i)$. Conversely, suppose $\mathbf{b} \in \{0, 1\}^{\tilde{E}}$ satisfies the proposed inequalities. For any $A \subset \tilde{E}$,

$$\begin{aligned} \text{rk}_{\tilde{P}}(A) &= \text{rk}_{\tilde{P}}(\bar{A}) = |\bar{A} \setminus \bar{A}^{\text{geo}}| + \text{rk}_{\tilde{P}}(\bar{A}^{\text{geo}}) && \text{by Lemma 2.4} \\ &= |\bar{A} \setminus \bar{A}^{\text{geo}}| + \text{rk}_P(\{i : E_i \subset \bar{A}\}) \\ &\geq |\bar{A} \setminus \bar{A}^{\text{geo}}| + \sum_{j \in \bar{A}^{\text{geo}}} b_j && \text{by hypothesis} \\ &\geq |\pi_{\bar{A}}(\mathbf{b})| \geq |\pi_A(\mathbf{b})|, && \text{so } \mathbf{b} \text{ is a basis of } \tilde{P}. \quad \square \end{aligned}$$

REMARK 2.9 (Realizing lifts). Let \mathbb{K} be a field and fix $\mathbf{n} \in \mathbb{Z}_{\geq 0}^N$. A linear subspace $V \subset \mathbb{K}^{\mathbf{n}}$ realizes both a polymatroid P on $E = \{1, \dots, N\}$ and a matroid M on $\tilde{E} = \{(i, j) : 1 \leq i \leq N, 1 \leq j \leq n_i\}$. Different $\prod_{i=1}^N \text{GL}(n_i)$ -translates of V may realize different matroids; however, the matroid of a general translate is \tilde{P} , and the polymatroid of any translate is P .

EXAMPLE 2.10. Suppose that $\mathbf{n} = (3)$ and that P is the polymatroid of rank 2 on one element, realized by a line $V \subset \mathbb{K}^{\mathbf{n}}$. Every line in $\mathbb{K}^{\mathbf{n}}$ realizes P because every line is codimension 2. However, different lines can have different matroids, e.g. a general line realizes a uniform matroid of rank 1 on 3 elements, while each of the three coordinate lines realizes a boolean matroid on 1 element with two loops adjoined.

In the remainder of this section, we describe how the matroid lift interacts with three operations on polymatroids. Throughout, we take (P, \mathbf{n}) to be a caged polymatroid on $E = \{1, \dots, N\}$, and \tilde{P} to be the multisymmetric lift of (P, \mathbf{n}) , on ground set $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$.

2.4. The **deletion** of $A \subset E$ is the polymatroid $P \setminus A$ obtained by restricting rk_P to subsets of $E \setminus A$. The **restriction** of P to A is $P|_A := P \setminus (E \setminus A)$. We were unable to find a reference for the following well-known description of the flats of $P \setminus A$.

LEMMA 2.11. *The flats of $P \setminus A$ are all sets of the form $F \setminus A$ such that F is a flat of P .*

Proof. A set $G \subset E \setminus A$ is a flat of $P \setminus A$ if and only if for all $i \in E \setminus A$, $\text{rk}_P(G \cup i) > \text{rk}_P(G)$. The latter condition is equivalent to saying the closure F of G in P contains no elements of $E \setminus A$, which is equivalent to saying $G = F \setminus A$. \square

Deletion commutes with lifts.

LEMMA 2.12. [9, Lemma 2.14⁽⁵⁾] For any subset $A \subset E$,

$$\widetilde{P \setminus A} = \widetilde{P} \setminus \cup_{i \in A} E_i,$$

where the left-hand lift is taken with respect to $\pi_{E \setminus A}(\mathbf{n})$.

REMARK 2.13 (Realizing deletions). With notation as in Example 2.2, if P is realized by V , then $P \setminus A$ is realized by $\pi_{E \setminus A}(V)$.

2.5. The **truncation** of P at a set $S \subset E$ is the polymatroid $T_S P$ on E , with rank function

$$\text{rk}_{T_S P}(A) := \begin{cases} \text{rk}_P(A) - 1, & \text{if } \text{rk}_P(A) = \text{rk}_P(A \cup S) \\ \text{rk}_P(A), & \text{otherwise.} \end{cases}$$

Since $\text{rk}_P(A \cup S) = \text{rk}_P(\overline{A \cup S}) = \text{rk}_P(A \cup \overline{S})$, $T_S P = T_{\overline{S}} P$.

LEMMA 2.14. If $S \subset E$, then

$$\widetilde{T_S P} = T_{\cup_{i \in S} E_i} \widetilde{P}.$$

Proof. Since $\cup_{i \in S} E_i$ is a geometric set of \widetilde{P} , the truncation $T_{\cup_{i \in S} E_i} \widetilde{P}$ is also a multisymmetric matroid on \widetilde{E} . For $A \subset E$,

$$\begin{aligned} \text{rk}_{T_{\cup_{i \in S} E_i} \widetilde{P}}(\cup_{i \in A} E_i) &= \begin{cases} \text{rk}_{\widetilde{P}}(\cup_{i \in A} E_i) - 1, & \text{if } \text{rk}_{\widetilde{P}}(\cup_{i \in A} E_i) = \text{rk}_{\widetilde{P}}(\cup_{i \in A \cup S} E_i) \\ \text{rk}_{\widetilde{P}}(\cup_{i \in A} E_i), & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{rk}_P(A) - 1, & \text{if } \text{rk}_P(A) = \text{rk}_P(A \cup S) \\ \text{rk}_P(A), & \text{otherwise} \end{cases} \\ &= \text{rk}_{T_S P}(A). \end{aligned}$$

Consequently, $\widetilde{T_S P} = T_{\cup_{i \in S} E_i} \widetilde{P}$ by Lemma 2.5. □

REMARK 2.15 (Realizing truncations). With notation as in Example 2.2, if $S \subset E$, then $T_S P$ is realized by $V \cap \pi_{E \setminus S}^{-1}(H)$, with H a general hyperplane in $\prod_{i \in S} W_i$.

2.6. The **reduction** of P at an element $i \in E$ of positive rank is the polymatroid $R_i P$ defined by

$$\text{rk}_{R_i P}(A) := \begin{cases} \text{rk}_P(A) - 1, & \text{if } \text{rk}_P(A) = \text{rk}_P(A \setminus i) + \text{rk}_P(i) \\ \text{rk}_P(A), & \text{otherwise.} \end{cases}$$

The **reduction** of a caged polymatroid is defined by

$$R_i(P, \mathbf{n}) := \begin{cases} (P, \mathbf{n} - \mathbf{e}_i), & n_i > \text{rk}_P(i) \\ (R_i P, \mathbf{n} - \mathbf{e}_i), & n_i = \text{rk}_P(i). \end{cases}$$

LEMMA 2.16. The multisymmetric lift of $R_i(P, \mathbf{n})$ is isomorphic to $\widetilde{P} \setminus j$, where j is any element of E_i .

Proof. We may think of both matroids as being multisymmetric on $\widetilde{E} \setminus j$. Geometric subsets of $\widetilde{E} \setminus j$ are of the form $(\cup_{k \in A} E_k) \setminus j$, with $A \subset E$. By Lemma 2.5 and Remark 2.6, it suffices to show that

$$\text{rk}_{\widetilde{P} \setminus j}((\cup_{k \in A} E_k) \setminus j) = \text{rk}_{R_i P}(A)$$

for all $A \subset E$. If $i \notin A$, then equality plainly holds, so we henceforth assume $i \in A$.

Suppose that $\text{rk}_P(A) = \text{rk}_P(A \setminus i) + \text{rk}_P(i)$. In this case,

$$\text{rk}_{\widetilde{P}}(A) \cup_{k \in A} E_k = \text{rk}_{\widetilde{P}}(\cup_{k \in A \setminus i} E_k) + \text{rk}_{\widetilde{P}}(E_i),$$

so every element of E_i is a coloop of $\tilde{P}|_{\cup_{k \in A} E_k}$. Hence,

$$\text{rk}_{\tilde{P} \setminus j}((\cup_{k \in A} E_k) \setminus j) = \text{rk}_{\tilde{P}}(\cup_{k \in A} E_k) - 1 = \text{rk}_P(A) - 1 = \text{rk}_{R_i P}(A).$$

Otherwise, suppose that $\text{rk}_P(A) < \text{rk}_P(A \setminus i) + \text{rk}_P(i)$. In this case, Lemma 2.4 implies that $(\cup_{k \in A} E_k) \setminus j \supset \cup_{k \in A} E_k$ in \tilde{P} . Hence,

$$\text{rk}_{\tilde{P} \setminus j}((\cup_{k \in A} E_k) \setminus j) = \text{rk}_{\tilde{P}}((\cup_{k \in A} E_k) \setminus j) = \text{rk}_{\tilde{P}}(\cup_{k \in A} E_k) = \text{rk}_P(A) = \text{rk}_{R_i P}(A). \quad \square$$

We defer discussion of realizing reductions to Remark 3.17, where we will see it in action en route to proving Theorem 1.1(ii).

3. COMBINATORIAL FLATS

We introduce the poset of combinatorial flats of a polymatroid P . We construct the poset using a chosen lift \tilde{P} in Section 3.1, then record the effects of deletion and truncation in Section 3.2. In Sections 3.3 and 3.4, we establish order-theoretic properties and consider the effects of reduction, culminating in a proof of Theorem 1.1. Finally, in Section 3.5, we axiomatize posets of combinatorial flats. This provides a cryptomorphic definition of simple polymatroids that generalizes the cryptomorphism of simple matroids and geometric lattices.

Our results on Schubert varieties of subspace arrangements depend only on Sections 3.1 and 3.2, so geometrically-inclined readers may go directly to Section 4 after reading these.

Throughout Section 3, we fix notations:

- (P, \mathbf{n}) is a caged polymatroid on $E = \{1, \dots, N\}$ with rank function rk_P ,
- \tilde{P} is the multisymmetric lift of (P, \mathbf{n}) , on ground set $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$, and
- $\text{rk}_P(\mathbf{s})$ is the rank of a multiset \mathbf{s} , in the sense of Section 1.2.

3.1. We construct the poset of combinatorial flats and note some properties of its rank function.

DEFINITION 3.1. A **combinatorial flat** of (P, \mathbf{n}) is a multiset $\mathbf{s} \in \mathbb{Z}_{\geq 0}^{\tilde{E}}$ such that there exists a flat F of \tilde{P} satisfying $\mathbf{s} = (|F \cap E_1|, \dots, |F \cap E_N|)$. (Rephrased in the language of Section 1.5, \mathbf{s} is a combinatorial flat if it is represented by a flat of \tilde{P} .)

Ordered by inclusion, the combinatorial flats of (P, \mathbf{n}) form the **poset of combinatorial flats**, denoted $\mathcal{L}_{P, \mathbf{n}}$. Observe that $\mathcal{L}_{P, \mathbf{n}}$ is isomorphic to the quotient poset $\mathcal{L}_{\tilde{P}}/\mathfrak{S}$, in which $\mathfrak{S} \cdot F \geq \mathfrak{S} \cdot F'$ if and only if there is $\sigma \in \mathfrak{S}$ such that $\sigma \cdot F \supset F'$. Since the action of \mathfrak{S} on $\mathcal{L}_{\tilde{P}}$ is rank-preserving, $\mathcal{L}_{P, \mathbf{n}}$ is a graded poset, and the rank of $\mathbf{s} \in \mathcal{L}_{P, \mathbf{n}}$ is equal to $\text{rk}_{\tilde{P}}(S)$, for any $S \in \mathcal{L}_{\tilde{P}}$ representing \mathbf{s} .

More generally, if $\mathbf{s} \leq \mathbf{n}$ is a multiset represented by $S \subset \tilde{E}$, then we may define its rank by $\text{rk}_{P, \mathbf{n}}(\mathbf{s}) := \text{rk}_{\tilde{P}}(S)$. The **closure** of \mathbf{s} is the combinatorial flat $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_N)$ represented by \bar{S} , and satisfies $\text{rk}_{P, \mathbf{n}}(\mathbf{s}) = \text{rk}_{P, \mathbf{n}}(\bar{\mathbf{s}})$. The **geometric part** of \mathbf{s} is \mathbf{s}^{geo} , the multiset represented by S^{geo} . Multisymmetry of \tilde{P} guarantees these definitions do not depend on choice of S .

Viewing $\mathcal{L}_{P, \mathbf{n}}$ as a quotient and applying Lemma 2.4, we learn that rank and closure of multisets have the following properties:

LEMMA 3.2.

- If \mathbf{s} is a combinatorial flat of (P, \mathbf{n}) , then $\text{rk}_{P, \mathbf{n}}(\mathbf{s}) = \text{rk}_{\mathbf{n}}(\mathbf{s}^{\text{geo}}) + |\mathbf{s} - \mathbf{s}^{\text{geo}}|$.
- If $\mathbf{a} \leq \mathbf{n}$ is a multiset, then for each $1 \leq i \leq N$, either $\bar{a}_i = a_i$ or $\bar{a}_i = n_i$.
- If \mathbf{s} is a combinatorial flat of (P, \mathbf{n}) and $0 < s_i < n_i$, then $\mathbf{s} - \mathbf{e}_i$ is also a combinatorial flat.

From submodularity of $\text{rk}_{\tilde{P}}$, we also obtain:

LEMMA 3.3. *If $\mathbf{s} \leq \mathbf{n} - \mathbf{e}_i$ is a multiset, then $\text{rk}_{P,n}(\mathbf{s} + \mathbf{e}_i) \leq \text{rk}_{P,n}(\mathbf{s}) + 1$.*

A **basis** of a multiset \mathbf{s} is an independent multiset of P maximal among those contained in \mathbf{s} .

LEMMA 3.4. *If \mathbf{b} is a basis of a multiset \mathbf{s} , then $\text{rk}_{P,n}(\mathbf{s}) = |\mathbf{b}|$. If \mathbf{s} is a combinatorial flat, then the bases of \mathbf{s} are precisely multisets of the form $\mathbf{b}' + \mathbf{s} - \mathbf{s}^{\text{geo}}$, where \mathbf{b}' is a basis of \mathbf{s}^{geo} .*

Proof. Choose $S \subset \tilde{E}$ representing \mathbf{s} . If \mathbf{b} is a basis of \mathbf{s} , then \mathbf{b} is represented by a basis B of S in \tilde{P} by Proposition 2.8. The first assertion holds because $\text{rk}_{P,n}(\mathbf{s}) = \text{rk}_{\tilde{P}}(S) = |B| = |\mathbf{b}|$.

If \mathbf{s} is a combinatorial flat, then $S \setminus S^{\text{geo}}$ consists of coloops of the flat S by Lemma 2.4. Hence, bases of S are of the form $B' \cup (S \setminus S^{\text{geo}})$ with B' a basis for S^{geo} . The second assertion follows. \square

REMARK 3.5. Lemma 3.4 shows that all bases of a multiset \mathbf{s} have the same cardinality, so $\text{rk}_P(\mathbf{s})$, as defined in Section 1.2, is well-defined.

REMARK 3.6 (Equivalence of definitions). Lemma 3.4 explains the equivalence of the different definitions of “rank” and “combinatorial flat” given in Section 1.2 and Section 3.1. The first statement of Lemma 3.4 says exactly that $\text{rk}_{P,n}(\mathbf{s}) = \text{rk}_P(\mathbf{s})$ for any multiset \mathbf{s} . When $\mathbf{s} \in \mathcal{L}_{P,n}$, this means $\text{rk}_P(\mathbf{s})$ is also equal to the rank of \mathbf{s} in the graded poset $\mathcal{L}_{P,n}$. We will henceforth write only $\text{rk}_P(\mathbf{s})$.

For equivalence of the two definitions of “combinatorial flat”, choose $S \subset \tilde{E}$ representing \mathbf{s} . If $s_i < n_i$, then $\mathbf{s} + \mathbf{e}_i$ is represented $S \cup e$ for any $e \in E_i \setminus S$. In the sense of Section 1.2, \mathbf{s} is a combinatorial flat if and only if

$$\text{rk}_{\tilde{P}}(S \cup e) = \text{rk}_{P,n}(\mathbf{s} + \mathbf{e}_i) = \text{rk}_P(\mathbf{s} + \mathbf{e}_i) > \text{rk}_P(\mathbf{s}) = \text{rk}_{P,n}(\mathbf{s}) = \text{rk}_{\tilde{P}}(S)$$

for all $e \in E_i$ with $s_i < n_i$. This is precisely the statement that S is a flat of \tilde{P} , which is the notion of “combinatorial flat” defined in Section 3.1.

REMARK 3.7 (Geometry of combinatorial flats). Suppose that the polymatroid P is realized by a subspace arrangement $V_1, \dots, V_N \subset V$. Rephrasing Remark 2.9 in the language of arrangements, if \mathbf{n} is a cage for P , then \tilde{P} is realized by any arrangement $\{H_{ij} : 1 \leq i \leq N, 1 \leq j \leq n_i\}$ in which H_{i1}, \dots, H_{in_i} are general hyperplanes containing V_i . The lattice of flats $\mathcal{L}_{\tilde{P}}$ has an action by a product of symmetric groups \mathfrak{S} , and the quotient $\mathcal{L}_{\tilde{P}}/\mathfrak{S}$ is isomorphic to \mathcal{L}_P (see also Section 3.1).

Alternatively, consider the subspace arrangement

$$\{V_{ij} : 1 \leq i \leq N, 1 \leq j \leq \text{codim } V_i\},$$

where $V_i = V_{i, \text{codim}_V V_i} \subset \dots \subset V_{i,2} \subset V_{i,1} \subsetneq V$ is a general flag. The poset of flats (not combinatorial flats) of the polymatroid associated to this arrangement is isomorphic to \mathcal{L}_P . If P is simple and $\mathbf{n} = (\text{rk}_P(1), \dots, \text{rk}_P(N))$, then the rank of $\mathbf{s} \in \mathcal{L}_{P,n}$ is $\text{rk}_P(\mathbf{s}) = \text{codim}_V V_{1,s_1} \cap \dots \cap V_{N,s_N}$.

REMARK 3.8. **Matricubes** were recently introduced by [1] to model intersection patterns of flags of linear subspaces. If (P, \mathbf{n}) is a caged polymatroid, then the function rk_P on multisets contained in \mathbf{n} defines a matricube. The **flats** [1, Definition 3.1] of this matricube are the combinatorial flats of P , and the **independents** [1, Definition 5.1] of the matricube are the independent multisets of P .

From the flag perspective of Remark 3.7, the difference between matricubes and combinatorial flats is that matricubes model *all* collections of flags, while combinatorial flats model only collections of *general* flags. Combinatorially, this difference

manifests in several ways: combinatorial flats obey Lemma 3.2, are top-heavy, and admit a reasonable notion of basis, but none of these properties hold for matricubes in general. (Failure of top-heaviness and Lemma 3.2 for matricubes can be seen in the right-hand example of [1, Section 3.1]. The difficulties of bases for matricubes are discussed in [1, Section 9.1].)

3.2. We record the effects deletion and truncation on $\mathcal{L}_{P,\mathbf{n}}$.

LEMMA 3.9. *Let $i \in E$. If $\mathbf{s} \leq \mathbf{n}$ is a multiset with $\bar{s}_i < n_i$, then $\pi_{E \setminus i}(\mathbf{s})$ is a combinatorial flat of $(P \setminus i, \pi_{E \setminus i}(\mathbf{n}))$ if and only if \mathbf{s} is a combinatorial flat of (P, \mathbf{n}) .*

Proof. The if direction follows from Lemma 2.11 and Lemma 2.12. For the “only if”, suppose that $\pi_{E \setminus i}(\mathbf{s})$ is a combinatorial flat of $(P \setminus i, \pi_{E \setminus i}(\mathbf{n}))$. We will show that \mathbf{s} must be a flat by proving $\mathbf{s} = \bar{\mathbf{s}}$. By Lemma 3.2 and the hypothesis that $\bar{s}_i < n_i$, it is always the case that $s_i = \bar{s}_i$. Hence, it suffices to show that $s_j = \bar{s}_j$ for $j \neq i$.

By Lemma 2.11 and Lemma 2.12, there is a combinatorial flat \mathbf{s}' of (P, \mathbf{n}) with $\pi_{E \setminus i}(\mathbf{s}') = \pi_{E \setminus i}(\mathbf{s})$. If $s'_i = n_i$, then $\mathbf{s} < \mathbf{s}'$, so $\mathbf{s} \leq \bar{\mathbf{s}} < \mathbf{s}'$. Entrywise, this implies $s_j \leq \bar{s}_j \leq s'_j = s_j$, which gives the desired equality.

Otherwise, $s'_i < n_i$. By Lemmas 3.2 and 3.3, $\mathbf{s}'' := \mathbf{s}' - s'_i \mathbf{e}_i = \mathbf{s} - s_i \mathbf{e}_i$ is a combinatorial flat of rank at least $\text{rk}(\mathbf{s}) - s_i$. By Lemma 3.2, $\bar{\mathbf{s}} - s_i \mathbf{e}_i$ is also a combinatorial flat, and has rank $\text{rk}(\bar{\mathbf{s}}) - s_i$. Since $\mathbf{s}'' \leq \bar{\mathbf{s}} - s_i \mathbf{e}_i$, these two combinatorial flats must be equal. In particular, for $j \neq i$, $s''_j = \bar{s}_j$. Since $s''_j = s_j$ by construction, this means $s_j = \bar{s}_j$, as desired. \square

LEMMA 3.10. *Let (P, \mathbf{n}) be a caged polymatroid on E . Let $F \subset E$ be a flat, and $\mathbf{f} = \sum_{i \in F} n_i \mathbf{e}_i$. The combinatorial flats of $(T_F P, \mathbf{n})$ are the combinatorial flats \mathbf{s} of P that either contain \mathbf{f} or satisfy $\bar{\mathbf{s}} + \mathbf{e}_i \not\geq \mathbf{f}$ for every $i \in E$ such that $s_i < n_i$. In the former case, $\text{rk}_{T_F P}(\mathbf{s}) = \text{rk}_P(\mathbf{s}) - 1$, and in the latter, $\text{rk}_{T_F P}(\mathbf{s}) = \text{rk}_P(\mathbf{s})$.*

Proof. For matroids and ordinary flats, this lemma is proved in [7, Proposition 7.4.9]. Explicitly, if P is a matroid, then the flats of $T_F P$ are the flats S of P that either contain F or satisfy $\overline{S \cup i} \not\supseteq F$ for every $i \in E \setminus S$.

By the matroid case and Lemma 2.14, a set S is a flat of $\widetilde{T_F P}$ if and only if S is a flat of \tilde{P} that either contains $\cup_{i \in F} E_i$ or satisfies $\overline{S \cup i} \not\supseteq \cup_{i \in F} E_i$ for all $i \in \tilde{E} \setminus S$. In the former case, $\text{rk}_{\widetilde{T_F P}}(S) = \text{rk}_{\tilde{P}}(S) - 1$; in the latter $\text{rk}_{\widetilde{T_F P}}(S) = \text{rk}_{\tilde{P}}(S)$. Taking \mathbf{s} to be the multiset represented by S completes the proof. \square

In the remainder of Section 3, we work up to proving Theorem 1.1, then provide axioms for combinatorial flats. These results are not required to understand Schubert varieties of subspace arrangements; readers interested solely in geometry may proceed to Section 4 forthwith.

3.3. We prove that $\mathcal{L}_{P,\mathbf{n}}$ is a top-heavy semimodular lattice, and identify its join-irreducibles in a special case. For multisets $\mathbf{s}, \mathbf{s}' \leq \mathbf{n}$, let $\mathbf{s} \vee \mathbf{s}' := (\max(s_i, s'_i))_i$ and $\mathbf{s} \wedge \mathbf{s}' := (\min(s_i, s'_i))_i$.

LEMMA 3.11. *If \mathbf{s} and \mathbf{s}' are multisets contained in \mathbf{n} , then*

$$\text{rk}_P(\mathbf{s} \wedge \mathbf{s}') + \text{rk}_P(\mathbf{s} \vee \mathbf{s}') \leq \text{rk}_P(\mathbf{s}) + \text{rk}_P(\mathbf{s}').$$

Proof. Pick sets $S, S' \subset \tilde{E}$ representing \mathbf{s} and \mathbf{s}' , respectively, such that $S \cap S'$ represents $\mathbf{s} \wedge \mathbf{s}'$ and $S \cup S'$ represents $(\max(s_i, s'_i))_i$. By submodularity of $\text{rk}_{\tilde{P}}$,

$$\begin{aligned} \text{rk}_P(\mathbf{s} \wedge \mathbf{s}') + \text{rk}_P(\mathbf{s} \vee \mathbf{s}') &= \text{rk}_{\tilde{P}}(S \cap S') + \text{rk}_{\tilde{P}}(\overline{S \cup S'}) = \text{rk}_{\tilde{P}}(S \cap S') + \text{rk}_{\tilde{P}}(S \cup S') \\ &\leq \text{rk}_{\tilde{P}}(S) + \text{rk}_{\tilde{P}}(S') = \text{rk}_P(\mathbf{s}) + \text{rk}_P(\mathbf{s}'). \end{aligned} \quad \square$$

LEMMA 3.12. $\mathcal{L}_{P,\mathbf{n}}$ is a semimodular lattice, with join and meet given by \vee and \wedge , respectively. Moreover, $\mathcal{L}_{P,\mathbf{n}}$ is top heavy and $|\mathcal{L}_{P,\mathbf{n}}^k| \leq |\mathcal{L}_{P,\mathbf{n}}^{k+1}|$ when $k \leq d/2$.

Proof. Let $\mathbf{s}, \mathbf{s}' \in \mathcal{L}_{P,\mathbf{n}}$. The join of \mathbf{s} and \mathbf{s}' is equal to $\mathbf{s} \vee \mathbf{s}'$ because any combinatorial flat containing \mathbf{s} and \mathbf{s}' must contain $(\max(s_i, s'_i))_i$, hence must also contain $\mathbf{s} \vee \mathbf{s}'$.

For meets: any combinatorial flat contained in both \mathbf{s} and \mathbf{s}' is contained in $\mathbf{s} \wedge \mathbf{s}'$, so we need only show that $\mathbf{s} \wedge \mathbf{s}'$ is a combinatorial flat. To see this, pick sets $S, S' \subset \tilde{E}$ representing \mathbf{s} and \mathbf{s}' , and such that $S \cap S'$ represents $\mathbf{s} \wedge \mathbf{s}'$. Since \mathbf{s} and \mathbf{s}' are combinatorial flats, S and S' are flats of \tilde{P} . Since flats are closed under intersection, $\mathbf{s} \wedge \mathbf{s}'$ is also a combinatorial flat. Hence, $\mathbf{s} \wedge \mathbf{s}'$ is the meet of \mathbf{s} and \mathbf{s}' in $\mathcal{L}_{P,\mathbf{n}}$.

Semimodularity is Lemma 3.11.

For top-heaviness: let $H^k(\tilde{P})$ be the \mathbb{Q} -vector space basis $\{v_F : F \in \mathcal{L}_{\tilde{P}}^k\}$, and let $H^k(P, \mathbf{n})$ be the \mathbb{Q} -vector space with basis $\{v_{\mathbf{s}} : \mathbf{s} \in \mathcal{L}_{P,\mathbf{n}}^k\}$. From the quotient construction of $\mathcal{L}_{P,\mathbf{n}}$, there are isomorphisms

$$H^k(P, \mathbf{n}) \xrightarrow{\cong} H^k(\tilde{P})^{\mathfrak{S}} \subset H^k(\tilde{P}), \quad v_{\mathbf{s}} \mapsto \sum_{\sigma \in \mathfrak{S}} v_{\sigma \cdot S},$$

where S is any flat representing \mathbf{s} . By [5, Theorem 1.1], there is an injective linear $L : H^k(\tilde{P}) \rightarrow H^{d-k}(\tilde{P})$ that commutes with the action of \mathfrak{S} . Consequently, the restriction of L to $H^k(\tilde{P})^{\mathfrak{S}}$ gives a linear injection

$$H^k(P, \mathbf{n}) \cong H^k(\tilde{P})^{\mathfrak{S}} \xrightarrow{L} H^{d-k}(\tilde{P})^{\mathfrak{S}} \cong H^{d-k}(P, \mathbf{n}),$$

and taking dimensions yields top-heaviness. Also by [5, Theorem 1.1], there are \mathfrak{S} -equivariant linear injections $H^0(\tilde{P}) \hookrightarrow H^1(\tilde{P}) \hookrightarrow \dots \hookrightarrow H^{\lfloor d/2 \rfloor}(\tilde{P})$, which restrict to linear injections

$$H^k(P, \mathbf{n}) \cong H^k(\tilde{P})^{\mathfrak{S}} \xrightarrow{L} H^{k+1}(\tilde{P})^{\mathfrak{S}} \cong H^{k+1}(P, \mathbf{n}),$$

which proves the remaining statement. □

Generalizations of Lemmas 3.11 and 3.12 for matricubes appear in [1, Theorem 3.4] (see Remark 3.8). With Lemmas 3.11 and 3.12 in hand, we have established Theorem 1.1 parts (i), (iii), and (iv). We now devote our attention to Theorem 1.1(ii), which states that P is determined by \mathcal{L}_P up to simplification.

An element of a lattice is **join-irreducible** if it cannot be written as a join of strictly smaller elements.

LEMMA 3.13. If P is simple and $\mathbf{n} = (\text{rk}_P(1), \dots, \text{rk}_P(N))$, then the join-irreducibles of $\mathcal{L}_{P,\mathbf{n}}$ are all multisets $s_i \mathbf{e}_i$, with $i \in E$ and $1 \leq s_i \leq n_i$.

Proof. Since P is simple, E_i is a flat of \tilde{P} for all $i \in E$. The restriction of \tilde{P} to E_i is a boolean matroid of rank $n_i = \text{rk}_P(i)$, so all proposed join-irreducibles are combinatorial flats of (P, \mathbf{n}) . Each $s_i \mathbf{e}_i$ is join-irreducible because the interval $[\mathbf{0}, s_i \mathbf{e}_i] \subset \mathcal{L}_{P,\mathbf{n}}$ is a chain of length s_i . No other combinatorial flats are join-irreducible because $\mathbf{s} \in \mathcal{L}_{P,\mathbf{n}}$ can be written $\mathbf{s} = s_1 \mathbf{e}_1 \vee \dots \vee s_N \mathbf{e}_N$. □

COROLLARY 3.14. If P is simple and $\mathbf{n} = (\text{rk}_P(1), \dots, \text{rk}_P(N))$, then P is determined up to relabelling of the ground set by the poset $\mathcal{L}_{P,\mathbf{n}}$.

Proof. By Lemma 3.13, each $i \in E$ corresponds to a maximal join-irreducible x_i of $\mathcal{L}_{P,\mathbf{n}}$. The rank function of P is recovered by $\text{rk}_P(S) = \text{rk}_{P,\mathbf{n}}(\vee_{i \in S} x_i)$. □

3.4. We show that reduction and loops do not change the lattice of combinatorial flats. From these facts, we derive Theorem 1.1.

LEMMA 3.15. *Suppose P is loopless. If $\{i\} \subset E$ is not a flat of P with rank n_i , then $\mathcal{L}_{R_i(P, \mathbf{n})} \cong \mathcal{L}_{P, \mathbf{n}}$.*

Proof. We will show the map $\psi : \mathcal{L}_{P, \mathbf{n}} \rightarrow \mathcal{L}_{R_i(P, \mathbf{n})}$ defined by

$$\psi(\mathbf{s}) = \begin{cases} \mathbf{s}, & s_i < n_i \\ \mathbf{s} - \mathbf{e}_i, & s_i = n_i \end{cases}$$

is an isomorphism. Multisets in the image of ψ are combinatorial flats of $R_i(\overline{P}, \mathbf{n})$ by Lemmas 2.16 and 2.11.

For surjectivity, suppose that $\mathbf{s} \in \mathcal{L}_{R_i(P, \mathbf{n})}$ is represented by a flat S of $\widetilde{R_i(\overline{P}, \mathbf{n})}$. We will show there is $\mathbf{s}' \in \mathcal{L}_{P, \mathbf{n}}$ with $\psi(\mathbf{s}') = \mathbf{s}$. By Lemmas 2.16 and 2.11, there is $e \in E_i \setminus S$ such that $S \cup e$ or S is a flat of \widetilde{P} . If S is a flat of \widetilde{P} , then $s_i < n_i$, so we may take $\mathbf{s}' = \mathbf{s}$. Otherwise, suppose $S \cup e$ is a flat of \widetilde{P} . If $s_i + 1 < n_i$, then by Lemma 3.2, \mathbf{s} is a combinatorial flat of \widetilde{P} , so we may again take $\mathbf{s}' = \mathbf{s}$. Otherwise, if $S \cup e$ is a flat of \widetilde{P} and $s_i + 1 = n_i$, then take $\mathbf{s}' = \mathbf{s} + \mathbf{e}_i$. This completes the proof of surjectivity.

We now prove injectivity. Let $\mathbf{s} \in \mathcal{L}_{R_i(P, \mathbf{n})}$. It is only possible for \mathbf{s} to have more than one preimage if $s_i = n_i - 1$. In this case, the two possible preimages of \mathbf{s} are \mathbf{s} and $\mathbf{s} + \mathbf{e}_i$.

If $\text{rk}_P(i) < n_i$, then \mathbf{s} cannot be a combinatorial flat of (P, \mathbf{n}) . This is because, assuming the contrary, we obtain the false inequality

$$\begin{aligned} \text{rk}(\mathbf{s}) < \text{rk}(\mathbf{s} + \mathbf{e}_i) &= \text{rk}((\mathbf{s} - (n_i - 1)\mathbf{e}_i) \vee n_i\mathbf{e}_i) \\ &\leq \text{rk}(\mathbf{s} - (n_i - 1)\mathbf{e}_i) + \text{rk}(n_i\mathbf{e}_i) \quad \text{by Lemma 3.11} \\ &= \text{rk}(\mathbf{s} - (n_i - 1)\mathbf{e}_i) + \text{rk}_P(i) \\ &\leq \text{rk}(\mathbf{s} - (n_i - 1)\mathbf{e}_i) + n_i - 1 \\ &= \text{rk}(\mathbf{s}) \quad \text{by Lemma 3.2 and the contrarian assumption.} \end{aligned}$$

Since \mathbf{s} is not a combinatorial flat of (P, \mathbf{n}) , \mathbf{s} has only one preimage when $\text{rk}_P(i) < n_i$.

Otherwise, we are in the case where $\text{rk}_P(i) = n_i$ and $\{i\}$ is not a flat of P . Choose sets $S \subset S' \subset \widetilde{E}$ representing \mathbf{s} and $\mathbf{s} + \mathbf{e}_i$, respectively, and assume towards a contradiction that S and S' are both flats of \widetilde{P} . Let e' be an element of $\overline{E_i} \setminus E_i$, which is nonempty because $\{i\}$ is not a flat of P . If e is the unique element of $S' \setminus S$, then S contains $\overline{(E_i \setminus e) \cup e'}$.

We now show that $\overline{(E_i \setminus e) \cup e'} = \overline{E_i}$, which implies that $S = S'$, a contradiction. By the definition of \widetilde{P} , there is $A \subset E$ such that

$$\text{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') = \text{rk}_P(A) + |((E_i \setminus e) \cup e') \setminus \cup_{k \in A} E_k|.$$

If $i \in A$, then $\text{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') \geq \text{rk}_P(i)$, so $\overline{(E_i \setminus e) \cup e'} = \overline{E_i}$. Otherwise, if $i \notin A$, then

$$\begin{aligned} \text{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') &= \text{rk}_P(A) + |E_i \setminus e| + |\{e'\} \setminus \cup_{k \in A} E_k| \\ &= \text{rk}_P(A) + \text{rk}_P(i) - 1 + |\{e'\} \setminus \cup_{k \in A} E_k|. \end{aligned}$$

This quantity is always at least $\text{rk}_P(i)$ because P is loopless; therefore, we have $\overline{(E_i \setminus e) \cup e'} = \overline{E_i}$. \square

LEMMA 3.16. *If i is a loop of P , then $\mathcal{L}_{P, \mathbf{n}} \cong \mathcal{L}_{P \setminus i, \pi_{E \setminus i}(\mathbf{n})}$.*

Proof. If i is a loop of P , every element of E_i is a loop of \tilde{P} , and the \mathfrak{S}_{E_i} factor of \mathfrak{S} acts trivially on $\mathcal{L}_{\tilde{P}}$. Hence,

$$\mathcal{L}_{P,\mathbf{n}} \cong \mathcal{L}_{\tilde{P}}/\mathfrak{S} \cong \mathcal{L}_{\tilde{P}\setminus i}/\pi_{E\setminus i}(\mathfrak{S}) \cong \mathcal{L}_{P\setminus i,\pi_{E\setminus i}(\mathbf{n})}. \quad \square$$

Proof of Theorem 1.1. Define $\mathcal{L}_P := \mathcal{L}_{P,\mathbf{n}}$, with \mathbf{n} any cage for P . This definition does not depend on the choice of cage because we may assume that P is loopless by Lemma 3.16, then repeatedly apply Lemma 3.15 to produce an isomorphism

$$\mathcal{L}_{P,\mathbf{n}} \cong \mathcal{L}_{P,(\text{rk}_P(1), \dots, \text{rk}_P(N))}.$$

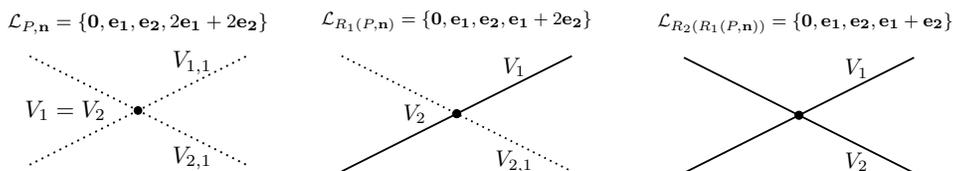
We will henceforth assume that $\mathbf{n} = (\text{rk}_P(1), \dots, \text{rk}_P(N))$.

Statements (1) and (4) are the content of Lemma 3.12; we now move to proving (2). To construct P^{sim} : let $(P', \mathbf{n}') = R_i(P \setminus L, \pi_{E \setminus L}(\mathbf{n}))$, where L is the set of loops of P and $i \in E \setminus L$ is any element such that $\{i\}$ is not a flat. If the resulting polymatroid P' is simple, then take $P^{\text{sim}} := P'$; otherwise, repeat these steps, replacing (P, \mathbf{n}) with (P', \mathbf{n}') . This process terminates because the cage shrinks at each step. If the cage is $\mathbf{0}$, then P is the polymatroid of rank 0 on zero elements, which is simple; otherwise, the process stopped before the cage was equal to $\mathbf{0}$, meaning that a simple polymatroid was produced.

Lemmas 3.15 and 3.16 show that $\mathcal{L}_{P^{\text{sim}}} \cong \mathcal{L}_P$, and Corollary 3.14 implies P^{sim} is the unique simple polymatroid with this property. This completes the proof of Theorem 1.1. \square

REMARK 3.17. If P is loopless and realized by a subspace arrangement $V_1, \dots, V_N \subset V$, then P fails to be simple if and only if $V_i \subset V_j \subsetneq V$ for some $i \neq j$. In this case, a step of the ‘‘simplification process’’ used to construct P^{sim} in the proof of Theorem 1.1 corresponds to replacing V_i with $V_i + \ell$, where $\ell \subset V$ is a generic line, then deleting loops. Alternatively, one may view this as replacing V_i with its immediate successor $V_{i, \text{codim } V_i - 1}$ in a general flag containing V_i (see Remark 3.7 and Example 3.18).

EXAMPLE 3.18. Suppose that P is realized by the subspace arrangement $V_1 = V_2 = 0 \subseteq V = \mathbb{C}^2$. Complete these subspaces to general flags $0 = V_1 \subsetneq V_{1,1} \subsetneq V$ and $0 = V_2 \subsetneq V_{2,1} \subsetneq V$, and let $\mathbf{n} = (2, 2)$. The steps of the reduction process for (P, \mathbf{n}) are illustrated below.



REMARK 3.19. If $j \in E$ is such that $\{j\}$ is a flat of P with positive rank, then $\{j\}$ is also a flat of P^{sim} . This fact follows from the claim below, which also provides a concrete way to see that the simplification process terminates.

Claim. If $\{j\}$ is a flat of a loopless polymatroid P and $i \in E \setminus j$, then either $\{i\}$ and $\{j\}$ are both flats of $R_i P$, or i is a loop of $R_i P$ and $\{i, j\}$ is a flat of $R_i P$.

Proof of Claim. Suppose i is a loop of $R_i P$. If $k \in E \setminus \{i, j\}$, then

$$\text{rk}_{R_i P}(\{i, j, k\}) = \text{rk}_{R_i P}(\{j, k\}) = \text{rk}_P(\{j, k\}) > \text{rk}_P(j) = \text{rk}_{R_i P}(\{i, j\}),$$

where the second equality holds because P is loopless. Hence, $\{i, j\}$ is a flat of $R_i P$.

Otherwise, suppose i is not a loop of $R_i P$. In this case, $\text{rk}_P(i) \geq 2$. If $k \in E \setminus i$ lies in the closure of $\{i\}$ in P , then

$$\text{rk}_{R_i P}(\{i, k\}) = \text{rk}_P(\{i, k\}) = \text{rk}_P(i) > \text{rk}_P(i) - 1 = \text{rk}_{R_i P}(i).$$

If k does not lie in the closure of $\{i\}$ in P , then

$$\text{rk}_{R_i P}(\{i, k\}) \geq \text{rk}_P(\{i, k\}) - 1 > \text{rk}_P(i) - 1 = \text{rk}_{R_i P}(i).$$

This shows that $\{i\}$ is a flat of $R_i P$. One similarly checks that $\{j\}$ remains a flat in $R_i P$. If $k \in E \setminus \{i, j\}$, then

$$\text{rk}_{R_i P}(\{j, k\}) = \text{rk}_P(\{j, k\}) > \text{rk}_P(j) = \text{rk}_{R_i P}(j).$$

If $\text{rk}_P(\{i, j\}) = \text{rk}_P(j) + \text{rk}_P(i)$, then

$$\text{rk}_{R_i P}(\{i, j\}) = \text{rk}_P(j) + \text{rk}_P(i) - 1 \geq \text{rk}_P(j) + 1 > \text{rk}_P(j) = \text{rk}_{R_i P}(j);$$

otherwise,

$$\text{rk}_{R_i P}(\{i, j\}) = \text{rk}_P(\{i, j\}) > \text{rk}_P(j) = \text{rk}_{R_i P}(j). \quad \square$$

3.5. We axiomatically characterize posets of combinatorial flats. In light of Theorem 1.1, this provides a cryptomorphic definition of simple polymatroids, generalizing the equivalence of geometric lattices and simple matroids.

Let \mathcal{L} be a graded lattice with minimum element $\hat{0}$. The **nullity** of $e \in \mathcal{L}$ is

$$\text{null}(e) = \#\{\text{join-irreducibles below } e\} - \text{rk}_{\mathcal{L}}(e),$$

where $\text{rk}_{\mathcal{L}}(e)$ is the length of any saturated chain in $[\hat{0}, e]$. The **geometric part** of e , denoted e^{geo} , is the join of all elements of $[\hat{0}, e]$ that are maximal irreducibles of \mathcal{L} .

THEOREM 3.20. *A graded poset \mathcal{L} is the lattice of combinatorial flats of a polymatroid P if and only if*

- \mathcal{L} is a semimodular lattice,
- the join-irreducibles of $\mathcal{L} \setminus \hat{0}$ form a downward-closed set, and
- for any $e \in \mathcal{L}$, $\text{null}(e) = \text{null}(e^{\text{geo}})$.

Proof. If P is a polymatroid (which we may assume simple by Theorem 1.1(2)), then the prescribed properties hold for \mathcal{L}_P by Theorem 1.1(1), Lemma 3.13, and Lemma 3.2.

Conversely, suppose that \mathcal{L} is a poset with the prescribed properties. Let E be the set of maximal join-irreducibles of \mathcal{L} and define a simple polymatroid P by $\text{rk}_P(A) := \text{rk}_{\mathcal{L}}(\vee_{i \in A} i)$. Let $\mathbf{n} = (\text{rk}_{\mathcal{L}}(i))_{i \in E}$.

Since the irreducibles of \mathcal{L} are downward closed, for each $i \in E$,

$$[\hat{0}, i] = \{\hat{0} < x_{i,1} < \cdots < x_{i,n_i} = i\}.$$

By construction of P , the following are equivalent:

- $F \subset E$ is a flat of rank r
- $\{i : x_{i,n_i} \leq \vee_{j \in F} x_{j,n_j}\} = F$ and $\text{rk}_{\mathcal{L}}(\vee_{j \in F} x_{j,n_j}) = r$,
- and $\sum_{j \in F} n_j \mathbf{e}_j$ is a combinatorial flat of rank r .

Define maps

$$\begin{aligned} \varphi : \mathcal{L}_P &\cong \mathcal{L}_{P,\mathbf{n}} \rightarrow \mathcal{L}, & \varphi(\mathbf{s}) &= \vee_{i \in E} x_{i,s_i} \\ \varphi' : \mathcal{L} &\rightarrow \mathcal{L}_{P,\mathbf{n}} \cong \mathcal{L}_P, & x &\mapsto \vee_{x_{i,j} \leq x} j \mathbf{e}_i. \end{aligned}$$

Plainly, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are increasing maps (i.e. $(\varphi \circ \varphi')(x) \geq x$ and $(\varphi' \circ \varphi)(\mathbf{s}) \geq \mathbf{s}$). To finish, we show that both φ and φ' decrease rank. Let \mathbf{s} be a combinatorial flat and let $F = \{i : s_i = n_i\}$, so that $\mathbf{s}^{\text{geo}} = \sum_{i \in F} n_i \mathbf{e}_i$. Using submodularity of $\text{rk}_{\mathcal{L}}$, our discussion of flats of P , and Lemma 3.2, we obtain

$$\begin{aligned} \text{rk}_{\mathcal{L}}(\varphi(\mathbf{s})) &\leq \text{rk}_{\mathcal{L}}(\vee_{i \in F} x_{i,n_i}) + \sum_{i \in E \setminus F} s_j \\ &= \text{rk}_P(F) + \sum_{i \in E \setminus F} s_j = \text{rk}_P(\mathbf{s}^{\text{geo}}) + |\mathbf{s} - \mathbf{s}^{\text{geo}}| = \text{rk}_P(\mathbf{s}). \end{aligned}$$

This shows that φ decreases rank. On the other hand, if $x \in \mathcal{L}$ and $F = \{i : x_{i,n_i} \leq x\}$, then

$$\sum_{i \in E} \max\{j : x_{ij} \leq x\} - \text{rk}_{\mathcal{L}}(x) = \text{null}(x) = \text{null}(x^{\text{geo}}) = \sum_{i: x_{i,n_i} \leq x} n_i - \text{rk}_{\mathcal{L}}(x^{\text{geo}}),$$

so

$$\begin{aligned} \text{rk}_{\mathcal{L}}(x) &= \text{rk}_{\mathcal{L}}(x^{\text{geo}}) + \sum_{i: x_{i,n_i} \not\leq x} \max\{j : x_{i,j} \leq x\} \\ &= \text{rk}_P \left(\sum_{i \in F} n_i \mathbf{e}_i \right) + \sum_{i: x_{i,n_i} \not\leq x} \max\{j : x_{i,j} \leq x\} \geq \text{rk}_P(\vee_{x_{i,j} \leq x} j \mathbf{e}_j) = \text{rk}_P(\varphi'(x)). \end{aligned}$$

This shows that φ' also decreases rank. Hence, both $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are increasing maps that decrease rank, which means they must be the identity. \square

4. SCHUBERT VARIETIES OF SUBSPACE ARRANGEMENTS

We introduce and study the Schubert variety $Y_V \subset \mathbb{P}^{\mathbf{n}}$ associated to a linear subspace $V \subset \mathbb{K}^{\mathbf{n}}$, with \mathbb{K} an algebraically closed field of characteristic 0. Our construction is founded on work of Hassett and Tschinkel [17], which we review in Sections 4.1 and 4.2. We construct Y_V in Section 4.4, then in Section 4.5 compute the support of its class in the Chow ring of $\mathbb{P}^{\mathbf{n}}$. In Section 4.6, we discuss “polymatroid generality”, a condition that guarantees Y_V consists of finitely many V -orbits. We assume this condition in Section 4.7, where we use the results of Section 4.5 to identify strata of Y_V , ending with a proof of Theorem 1.3. Finally, in Section 4.8, we give a presentation for the singular cohomology ring of Y_V when $\mathbb{K} = \mathbb{C}$ and V is polymatroid general.

4.1. We explain a construction of Hassett and Tschinkel [17], which we will eventually think of as the variety associated to a polymatroid on one element. Loosely speaking, Hassett and Tschinkel show that there is only one action of the n -dimensional additive group \mathbb{G}_a^n on \mathbb{P}^n with finitely many orbits. More precisely:

THEOREM 4.1 ([17, Proposition 3.7]). *Over an algebraically closed field of characteristic 0, there is a left action of \mathbb{G}_a^n on \mathbb{P}^n with finitely many orbits. The action is unique up to isomorphism of \mathbb{G}_a^n -variety structures.*

To make matters concrete, choose coordinates a_1, \dots, a_n and b_0, \dots, b_n on \mathbb{G}_a^n and \mathbb{P}^n , respectively. The action on \mathbb{P}^n is given by the faithful representation $\rho_n : \mathbb{G}_a^n \rightarrow \text{Aut}(\mathbb{P}^n) = \text{PGL}_{n+1}$,

$$\mathbf{a} = (a_1, \dots, a_n) \mapsto \exp \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & \ddots & \vdots \\ a_2 & a_1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_n & \cdots & a_2 & a_1 & 0 \end{pmatrix}.$$

The entries in each northwest-to-southeast diagonal of $\rho_n(\mathbf{a})$ are all equal to one another. Explicitly, the (i, j) entry is

$$(\rho_n(\mathbf{a}))_{i,j} = \begin{cases} \frac{1}{(i-j)!} B_{i-j}(1!a_1, 2!a_2, \dots, (i-j)!a_{i-j}), & i \geq j \\ 0, & \text{otherwise,} \end{cases}$$

where B_k is the (exponential) **Bell polynomial** of degree k , defined by $B_0 = 1$ and

$$B_i(x_1, \dots, x_i) = i! \sum_{j_1+2j_2+\dots+ij_i=i} \frac{1}{j_1!j_2!\dots j_i!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_i}{i!}\right)^{j_i} \quad \text{for } i \geq 1.$$

This description implies the following facts.

PROPOSITION 4.2. *The action of \mathbb{G}_a^n on \mathbb{P}^n partitions \mathbb{P}^n into orbits*

$$O_k := \{b_0 = b_1 = \dots = b_{n-k-1} = 0, b_{n-k} \neq 0\}, \quad 0 \leq k \leq n.$$

The stabilizer of any point of O_k is

$$\text{St}_k := \{a_1 = a_2 = \dots = a_k = 0\} \subset \mathbb{G}_a^n,$$

and the closure of O_k in \mathbb{P}^n is $\overline{O_k} = \cup_{i \leq k} O_i$.

EXAMPLE 4.3. The action of \mathbb{G}_a^2 on \mathbb{P}^2 is

$$\begin{aligned} (a_1, a_2) \cdot [b_0 : b_1 : b_2] &= \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ \frac{1}{2}a_1^2 + a_2 & a_1 & 1 \end{pmatrix} \cdot [b_0 : b_1 : b_2] \\ &= [b_0 : a_1b_0 + b_1 : \frac{1}{2}a_1^2b_0 + a_2b_0 + a_1b_1 + b_2]. \end{aligned}$$

4.2. Consider now the semidirect product $\mathbb{G}_a^n \rtimes \mathbb{G}_m$ (with $n \geq 1$), in which \mathbb{G}_m acts on \mathbb{G}_a^n by $t \cdot \mathbf{a} = (ta_1, t^2a_2, \dots, t^na_n)$. If we define

$$\lambda_n : \mathbb{G}_m \rightarrow \text{Aut}(\mathbb{P}^n), \quad t \mapsto \begin{pmatrix} 1 & & & \\ & t & & \\ & & t^2 & \\ & & & \ddots \\ & & & & t^n \end{pmatrix} \in \text{Aut}(\mathbb{P}^n),$$

then ρ_n extends to an injective homomorphism

$$\mathbb{G}_a^n \rtimes \mathbb{G}_m \hookrightarrow \text{Aut}(\mathbb{P}^n), \quad (\mathbf{a}, t) \mapsto \rho_n(\mathbf{a})\lambda_n(t),$$

in which the image of λ_n normalizes the image of ρ_n . The orbits of $\mathbb{G}_a^n \rtimes \mathbb{G}_m$ on \mathbb{P}^n are the same as those of \mathbb{G}_a^n , but each \mathbb{G}_a^n -orbit O_i contains a unique \mathbb{G}_m -fixed point. In coordinates, the fixed point is $[\mathbf{e}_{n+1-i}]$. Consequently, we may canonically identify O_i with $\mathbb{G}_a^n/\text{St}_i$ via the map $\mathbf{a} \mapsto \rho_n(\mathbf{a}) \cdot [\mathbf{e}_{n+1-i}]$.

4.3. Throughout the remainder of Section 4, fix notation:

- $V \subset \mathbb{K}^{\mathbf{n}} = \mathbb{K}^{n_1} \times \dots \times \mathbb{K}^{n_N}$ is a d -dimensional linear subspace,
- a_{ij} and b_{ik} , with $1 \leq i \leq N$, $1 \leq j \leq n_i$, and $0 \leq k \leq n_i$, are coordinates on $\mathbb{K}^{\mathbf{n}}$ and $\mathbb{P}^{\mathbf{n}}$, respectively,
- P is the polymatroid on $E = \{1, \dots, N\}$ associated to V ,
- \tilde{P} is the lift of P with respect to \mathbf{n} , on ground set $\tilde{E} = E_1 \sqcup \dots \sqcup E_N$.

Phrases like “combinatorial flat of V ” should be taken to mean “combinatorial flat of (P, \mathbf{n}) .” In Sections 4.7 and 4.8, we will assume that V is “polymatroid general”, in the sense defined in Section 4.6.

4.4. We construct Schubert varieties of subspace arrangements, bootstrapping from the Hassett-Tschinkel action of Section 4.1. Identifying $\mathbb{G}_a^{n_i}$ and \mathbb{K}^{n_i} , the Hassett-Tschinkel maps $\rho_{n_i} : \mathbb{K}^{n_i} \rightarrow \text{Aut}(\mathbb{P}^{n_i})$ assemble to a map $\rho : \mathbb{K}^{\mathbf{n}} \rightarrow \text{Aut}(\mathbb{P}^{\mathbf{n}})$, allowing $\mathbb{K}^{\mathbf{n}}$ to act on $\mathbb{P}^{\mathbf{n}}$. For each multiset $\mathbf{s} \leq \mathbf{n}$, there is an $|\mathbf{s}|$ -dimensional orbit

$$O_{\mathbf{s}} := O_{s_1} \times \dots \times O_{s_N} \subset \mathbb{P}^{\mathbf{n}},$$

whose closure is a product of projectivized coordinate subspaces

$$\mathbb{P}^{\mathbf{s}} := \mathbb{P}^{s_1} \times \dots \times \mathbb{P}^{s_N} \subset \mathbb{P}^{\mathbf{n}}.$$

All points in $O_{\mathbf{s}}$ have the same stabilizer, denoted by $\text{St}_{\mathbf{s}}$. The explicit description in Section 4.1 shows that $\text{St}_{\mathbf{s}}$ is the coordinate subspace of $\mathbb{K}^{\mathbf{n}}$ cut out by

$$a_{i,1} = a_{i,2} = \dots = a_{i,s_i} = 0, \quad 1 \leq i \leq N.$$

By Section 4.2, ρ extends to an injection

$$\mathbb{K}^{\mathbf{n}} \rtimes \mathbb{G}_m \hookrightarrow \text{Aut}(\mathbb{P}^{\mathbf{n}}),$$

and each orbit $O_{\mathbf{s}}$ contains a unique \mathbb{G}_m -fixed point $([\mathbf{e}_{n_1+1-s_1}], \dots, [\mathbf{e}_{n_N+1-s_N}])$. This yields a canonical isomorphism

$$\iota : \mathbb{K}^{\mathbf{n}} \rightarrow O_{\mathbf{n}}, \quad \mathbf{a} \mapsto \rho(\mathbf{a}) \cdot ([\mathbf{e}_1], \dots, [\mathbf{e}_1]).$$

DEFINITION 4.4. The **Schubert variety** of $V \subset \mathbb{K}^{\mathbf{n}}$ is $Y_V := \overline{\iota(V)}$, the closure of $\iota(V)$ in $\mathbb{P}^{\mathbf{n}}$.

4.5. We compute the support of the Chow class of Y_V . Given a nonsingular variety X , let $A_d(X)$ be the Chow group of X spanned by d -dimensional cycles. If Y and Y' are subvarieties of X , then their intersection product is written $[Y] \cdot [Y']$.

LEMMA 4.5. The Chow class of Y_V in $A_d(\mathbb{P}^{\mathbf{n}})$ is of the form

$$[Y_V] = \sum_{\mathbf{b} \text{ basis of } P} c_{\mathbf{b}} [\mathbb{P}^{\mathbf{b}}].$$

Proof. Write $[Y_V] = \sum_{|\mathbf{t}|=d} c_{\mathbf{t}} [\mathbb{P}^{\mathbf{t}}]$. Let \mathbf{s} be a multiset with $|\mathbf{s}| = d$ that is not a basis of P . We will show that $[\mathbb{P}^{\mathbf{s}}]$ has coefficient zero in $[Y_V]$.

Let $T = \{i : \bar{s}_i = n_i\}$. This set is nonempty: if we suppose falsely that $T = \emptyset$, then by Lemma 3.2, $\mathbf{s} = \bar{\mathbf{s}}$. This means that \mathbf{s} is a combinatorial flat satisfying $\mathbf{s}^{\text{geo}} = \mathbf{0}$, so by Lemma 3.2 once more,

$$\text{rk}_P(\mathbf{s}) = \text{rk}_P(\mathbf{s}^{\text{geo}}) + |\mathbf{s} - \mathbf{s}^{\text{geo}}| = 0 + |\mathbf{s}| = d,$$

contrary to the fact that \mathbf{s} is not a basis.

Let $\mathbf{s}' = \mathbf{s} - \sum_{i \in E \setminus T} s_i \mathbf{e}_i$. The pushforward $\pi_{T*} : A_0(\mathbb{P}^{\mathbf{n}}) \rightarrow A_0(\mathbb{P}^{\pi_T(\mathbf{n})})$ is an isomorphism, so $c_{\mathbf{s}} = 0$ if and only if $\pi_{T*}(c_{\mathbf{s}}[\text{pt}])$ is zero in $A_*(\mathbb{P}^{\pi_T(\mathbf{n})})$. By the projection formula,

$$\begin{aligned} \pi_{T*}(c_{\mathbf{s}}[\text{pt}]) &= \pi_{T*}([\mathbb{P}^{\mathbf{n}-\mathbf{s}}] \cdot [Y_V]) = \pi_{T*}([\mathbb{P}^{\mathbf{n}-\mathbf{s}'}] \cdot [\mathbb{P}^{\mathbf{n}-\mathbf{s}+\mathbf{s}'}] \cdot [Y_V]) \\ &= [\mathbb{P}^{\pi_T(\mathbf{n}-\mathbf{s}')}] \cdot \pi_{T*}([\mathbb{P}^{\mathbf{n}-\mathbf{s}+\mathbf{s}'}] \cdot [Y_V]). \end{aligned}$$

The class $[\mathbb{P}^{\mathbf{n}-\mathbf{s}+\mathbf{s}'}] \cdot [Y_V]$ is represented by a subvariety Y' of Y_V , obtained by intersecting Y_V with appropriately general hyperplanes. To show that $[\mathbb{P}^{\pi_T(\mathbf{n}-\mathbf{s}')}] \cdot \pi_{T*}([\mathbb{P}^{\mathbf{n}-\mathbf{s}+\mathbf{s}'}] \cdot [Y_V]) = [\mathbb{P}^{\pi_T(\mathbf{n}-\mathbf{s}')}] \cdot \pi_{T*}([Y'])$ is zero, it suffices to show that $\dim \pi_T(Y') < |\pi_T(\mathbf{s}')|$. In fact, this is the case: by Lemma 3.2,

$$\begin{aligned} \dim \pi_T(Y') &\leq \dim \pi_T(Y_V) = \text{rk}_P(T) = \text{rk}_{P,\mathbf{n}}(\bar{\mathbf{s}}^{\text{geo}}) \\ &= \text{rk}_{P,\mathbf{n}}(\bar{\mathbf{s}}) - \sum_{i \in E \setminus T} s_i < d - \sum_{i \in E \setminus T} s_i = \sum_{i \in T} s_i = |\pi_T(\mathbf{s}')|. \end{aligned}$$

We conclude that $c_{\mathbf{s}} = 0$. □

LEMMA 4.6. For each basis \mathbf{b} of P , $c_{\mathbf{b}} > 0$.

Proof. Let $I : \mathbb{K}^{\mathbf{n}} \rightarrow O_{\mathbf{n}} \subset \mathbb{P}^{\mathbf{n}}$ be the usual map from $\prod_{i=1}^N \mathbb{K}^{n_i}$ to $\prod_{i=1}^N \mathbb{P}^{n_i}$, given by $\frac{b_{ij}}{b_{i0}}(I(\mathbf{a})) = a_{ij}$. Write \bar{V} for the closure of $I(V)$ in $\mathbb{P}^{\mathbf{n}}$. The Chow class of \bar{V} is (see e.g. [8]) given by

$$[\bar{V}] = \sum_{\mathbf{b} \text{ basis of } P} [\mathbb{P}^{\mathbf{b}}].$$

Therefore if we can find a flat family X_t of subvarieties of $\mathbb{P}^{\mathbf{n}}$ such that X_1 is isomorphic to Y_V and $X_0 \supset \bar{V}$, then $c_{\mathbf{b}} \geq 1$ for all bases \mathbf{b} of P . Recall from Section 4.4 that Y_V is the closure of $\iota(V)$, where $\iota : \mathbb{K}^{\mathbf{n}} \rightarrow O_{\mathbf{n}}$. The idea is that $I(V)$ is the tangent space to $\iota(V)$ at the origin of $O_{\mathbf{n}}$, so we can construct a deformation of Y_V by taking the closure of the deformation of $\iota(V)$ to its tangent space.

We now construct such a family. Allow \mathbb{G}_m to act on $\mathbb{K}^{\mathbf{n}}$ and $\mathbb{P}^{\mathbf{n}}$ in the standard way, given in the i th factor by

$$\begin{aligned} t \cdot (a_{i,1}, \dots, a_{i,n_i}) &= (ta_{i,1}, \dots, ta_{i,n_i}) \quad \text{and} \\ t \cdot [b_{i,0} : b_{i,1} : \dots : b_{i,n_i}] &= [b_{i,0} : tb_{i,1} : \dots : tb_{i,n_i}], \end{aligned}$$

and define

$$\iota'_0 : \mathbb{K}^{\mathbf{n}} \times \mathbb{G}_m \rightarrow O_{\mathbf{n}} \times \mathbb{A}^1, \quad (\mathbf{a}, t) \mapsto t^{-1} \cdot \iota(t \cdot \mathbf{a}).$$

From Section 4.1, we see that

$$\frac{b_{i\ell}}{b_{i0}}(\iota'_0(\mathbf{a}, t)) = \sum_{j_1+2j_2+\dots+\ell j_{\ell}=\ell} t^{j_1+j_2+\dots+j_{\ell}-1} \frac{a_{i,1}^{j_1} a_{i,2}^{j_2} \dots a_{i,\ell}^{j_{\ell}}}{j_1! j_2! \dots j_{\ell}!}, \quad 1 \leq \ell \leq n_i,$$

so ι'_0 extends to a regular map

$$\iota' : \mathbb{K}^{\mathbf{n}} \times \mathbb{A}^1 \rightarrow O_{\mathbf{n}} \times \mathbb{A}^1$$

satisfying

$$\frac{b_{i\ell}}{b_{i0}}(\iota'(\mathbf{a}, 0)) = a_{i\ell}.$$

Let X be the closure of $\iota'(V \times \mathbb{A}^1)$ in $\mathbb{P}^{\mathbf{n}} \times \mathbb{A}^1$. Equivalently, X is the closure of $\iota'_0(V \times \mathbb{G}_m)$, so X is a flat family over \mathbb{A}^1 (e.g. by [20, Proposition 4.3.9]).

Observe that $X_0 \supset \bar{V}$. On the other hand, there is a diagram

$$\begin{array}{ccc} V \times \mathbb{G}_m & \hookrightarrow & \mathbb{K}^{\mathbf{n}} \times \mathbb{G}_m \xrightarrow{\iota' \times \text{Id}} \mathbb{P}^{\mathbf{n}} \times \mathbb{G}_m \\ \downarrow \cong & & \downarrow \cong \\ V \times \mathbb{G}_m & \hookrightarrow & \mathbb{K}^{\mathbf{n}} \times \mathbb{G}_m \xrightarrow{\iota \times \text{Id}} \mathbb{P}^{\mathbf{n}} \times \mathbb{G}_m \end{array} \quad \text{given fiberwise as} \quad \begin{array}{ccc} V & \hookrightarrow & \mathbb{K}_t^{\mathbf{n}} \xrightarrow{\iota'} \mathbb{P}_t^{\mathbf{n}} \\ \downarrow t. & & \downarrow t. \\ V & \hookrightarrow & \mathbb{K}_t^{\mathbf{n}} \xrightarrow{\iota} \mathbb{P}_t^{\mathbf{n}}. \end{array}$$

Taking closures, we obtain an isomorphism of families

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^{\mathbf{n}} \times \mathbb{G}_m \\ \downarrow \cong & & \downarrow \cong \\ Y_V \times \mathbb{G}_m & \hookrightarrow & \mathbb{P}^{\mathbf{n}} \times \mathbb{G}_m, \end{array}$$

that identifies X_1 with Y_V . □

REMARK 4.7. Another proof of Lemma 4.6 goes as follows: a projection of $O_{\mathbf{n}} \cap Y_V$ onto a given d -element subset of the coordinates $\{b_{ij}/b_{i0}\}_{i,j}$ is dominant if and only if its induced map on tangent spaces is surjective at a general point of $O_{\mathbf{n}} \cap Y_V$. Consequently, to prove that $c_{\mathbf{b}} > 0$, it suffices to produce a point $p \in O_{\mathbf{n}} \cap Y_V$ and a d -element set of coordinates representing \mathbf{b} such that the derivative of the corresponding projection is surjective at p .

The derivative of ι at $0 \in V$ identifies V with the tangent space of Y_V at the point $p = [1 : 0 : 0 : \dots : 0] \times \dots \times [1 : 0 : 0 : \dots : 0]$. Therefore, if \mathbf{b} is a basis of

the polymatroid P associated to V , then a d -element set of the type desired exists, proving that $c_{\mathbf{b}} > 0$.

4.6. The Schubert variety Y_V is V -equivariant, hence is a union of V -orbits. To guarantee that the number of orbits is finite, we require V to be **polymatroid general** (henceforth, **p.g.**), meaning that for each multiset $\mathbf{s} \leq \mathbf{n}$,

$$\text{rk}_{P,\mathbf{n}}(\mathbf{s}) = \text{codim}_V V \cap \text{St}_{\mathbf{s}}.$$

If the matroid of V in $(\mathbb{K}^1)^{|\mathbf{n}|} \cong \mathbb{K}^{\mathbf{n}}$ is \tilde{P} , then V is polymatroid general, so any realizable polymatroid has a p.g. realization by Remark 2.9. However, not all p.g. realizations of P realize \tilde{P} .

The next two results allow us to make inductive arguments; we will use them often without mention.

PROPOSITION 4.8. *If V is p.g. and $i \in E$, then $\pi_{E \setminus i}(V)$ is p.g.*

Proof. Let $\mathbf{s}' \leq \pi_{E \setminus i}(\mathbf{n})$ be a multiset and let $\mathbf{s} = \sum_{j \in E \setminus i} s'_j \mathbf{e}_j$. There is a commuting diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V \cap \ker(\pi_{E \setminus i}) & \longrightarrow & V \cap \text{St}_{\mathbf{s}} & \longrightarrow & V' \cap \text{St}_{\mathbf{s}'} & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V \cap \ker(\pi_{E \setminus i}) & \longrightarrow & V & \xrightarrow{\pi_{E \setminus i}} & V' & \longrightarrow & 0, \end{array}$$

so $\text{codim}_{V'} V' \cap \text{St}_{\mathbf{s}'} = \text{codim}_V V \cap \text{St}_{\mathbf{s}} = \text{rk}_P(\mathbf{s})$. The multisets \mathbf{s} and \mathbf{s}' are represented by the same set $S \subset \tilde{E} \setminus E_i$, so

$$\text{codim}_{V'} V' \cap \text{St}_{\mathbf{s}'} = \text{rk}_{P,\mathbf{n}}(\mathbf{s}) = \text{rk}_{\tilde{P}}(S) = \text{rk}_{\tilde{P} \setminus E_i}(S) = \text{rk}_{\widetilde{P \setminus i}}(S) = \text{rk}_{P \setminus i, \pi_{E \setminus i}(\mathbf{n})}(\mathbf{s}'),$$

which is the desired conclusion. □

PROPOSITION 4.9. *Suppose $V \subset \mathbb{K}^{\mathbf{n}}$ is a p.g. subspace and $A \subset E$. If H is the preimage in $\mathbb{K}^{\mathbf{n}}$ of a general⁽⁶⁾ hyperplane in $\mathbb{K}^{\tau_A(\mathbf{n})}$, then $V \cap H$ is p.g.*

Proof. Let $\mathbf{s} \leq \mathbf{n}$ be a multiset represented by $S \subset \tilde{E}$. From Section 2.5, we learn

$$\begin{aligned} \text{rk}_{T_A P, \mathbf{n}}(\mathbf{s}) &= \text{rk}_{\widetilde{T_A P}}(S) = \begin{cases} \text{rk}_{\tilde{P}}(S) - 1, & \text{if } \cup_{i \in A} E_i \subset \bar{S} \\ \text{rk}_{\tilde{P}}(S), & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{rk}_{P,\mathbf{n}}(\mathbf{s}) - 1, & \text{if } A \subset \{i : \bar{s}_i = n_i\} \\ \text{rk}_{P,\mathbf{n}}(\mathbf{s}), & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{codim}_{V \cap H} V \cap H \cap \text{St}_{\mathbf{s}} &= \begin{cases} \text{codim}_V V \cap \text{St}_{\mathbf{s}} - 1, & \text{if } V \cap \text{St}_{\mathbf{s}} \subset V \cap H \\ \text{codim}_V V \cap \text{St}_{\mathbf{s}}, & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{rk}_{P,\mathbf{n}}(\mathbf{s}) - 1, & \text{if } V \cap \text{St}_{\mathbf{s}} \subset V \cap H \\ \text{rk}_{P,\mathbf{n}}(\mathbf{s}), & \text{otherwise.} \end{cases} \end{aligned}$$

To finish, observe that $A \subset \{i : \bar{s}_i = n_i\}$ if and only if $V \cap \text{St}_{\mathbf{s}} \subset V \cap H$ for a Zariski-general H . □

⁽⁶⁾By which we mean a hyperplane such that $H \cap V$ realizes $T_A P$. See also Remark 2.15.

4.7. We establish Theorem 1.3 via a series of lemmas characterizing the strata of Y_V . Throughout this section, V is assumed to be polymatroid general.

LEMMA 4.10. *If \mathbf{s} is not a combinatorial flat of V , then $Y_V \cap O_{\mathbf{s}}$ is empty.*

Proof. We induct on N . For any point $x \in O_{\mathbf{s}}$,

$$(*) \quad \dim(V \cdot x) = \text{codim}_V(V \cap \text{St}_{\mathbf{s}}) = \text{rk}_P(\mathbf{s}).$$

Since Y_V is a union of V -orbits and V is p.g., the desired statement follows immediately from dimensional considerations when $N = 1$.

Otherwise, suppose $N > 1$. If $\text{rk}(\mathbf{s}) = d$, then the dimension formula implies $Y_V \cap O_{\mathbf{s}}$ is nonempty if and only if $\mathbf{s} = \mathbf{n}$. Otherwise, if $\text{rk}(\mathbf{s}) < d$, then there is $i \in E$ such that $\bar{s}_i < n_i$. The projection $Y_V \rightarrow Y_{\pi_{E \setminus i}(V)}$ sends $Y_V \cap O_{\mathbf{s}}$ into $Y_{\pi_{E \setminus i}(V)} \cap O_{\pi_{E \setminus i}(\mathbf{s})}$. By Lemma 3.9, $\pi_{E \setminus i}(\mathbf{s})$ is not a combinatorial flat of $\pi_{E \setminus i}(V)$, so $Y_{\pi_{E \setminus i}(V)} \cap O_{\pi_{E \setminus i}(\mathbf{s})}$ is empty by induction on N . This means $Y_V \cap O_{\mathbf{s}}$ is empty too. \square

The following result gives a sharp description of the geometry of a nonempty stratum.

LEMMA 4.11. *If \mathbf{s} is a combinatorial flat of V such that $Y_V \cap O_{\mathbf{s}}$ is nonempty and $V' = \pi_{\{i: s_i = n_i\}}(V)$, then $Y_V \cap O_{\mathbf{s}} = \iota(V') \times \prod_{s_i < n_i} O_{s_i}$. Consequently, $Y_V \cap O_{\mathbf{s}}$ is a single V -orbit of dimension $\text{rk}(\mathbf{s})$, and*

$$\overline{Y_V \cap O_{\mathbf{s}}} = Y_{V'} \times \prod_{s_i < n_i} \mathbb{P}^{s_i} = Y_{V' \times \prod_{s_i < n_i} \mathbb{K}^{s_i}} \subset \mathbb{P}^{\mathbf{s}}.$$

Proof. The statement plainly holds when $\mathbf{s} = \mathbf{n}$. Otherwise, assume \mathbf{s} is a proper combinatorial flat. The V -orbit of any point in $O_{\mathbf{s}}$ has dimension $\text{rk}(\mathbf{s})$ and Y_V is V -equivariant, so $Y_V \cap O_{\mathbf{s}}$ has dimension at least $\text{rk}(\mathbf{s})$.

We complete the proof for proper combinatorial flats by inducting on N . First consider $N = 1$, so that $Y_V \subset \mathbb{P}^{n_1}$. By Lemma 4.10, $Y_V \cap \mathbb{P}^{n_1-1} \subset \mathbb{P}^{d \dim V - 1}$. Since $Y_V \cap \mathbb{P}^{n_1-1}$ is a divisor on Y_V , we must have $Y_V \cap \mathbb{P}^{n_1-1} = \overline{\mathbb{P}^{d \dim V - 1}}$. Since V is polymatroid general, $O_i \subset \mathbb{P}^{d \dim V - 1}$ is a single V -orbit, and $\overline{Y_V \cap O_i} = \mathbb{P}^i$ as desired.

Now, suppose $N > 1$. Since \mathbf{s} is a proper combinatorial flat, there is $j \in E$ such that $s_j < n_j$. Set $V' = \pi_{E \setminus j}(V)$ and $\mathbf{s}' = \pi_{E \setminus j}(\mathbf{s})$. The projection $\pi_{E \setminus j} : \mathbb{P}^{\mathbf{n}} \rightarrow \mathbb{P}^{\pi_{E \setminus j}(\mathbf{n})}$ induces a surjection $Y_V \rightarrow Y_{V'}$. This map sends $Y_V \cap O_{\mathbf{s}}$ into $Y_{V'} \cap O_{\mathbf{s}'}$, so by the induction hypothesis

$$\begin{aligned} Y_V \cap O_{\mathbf{s}} &\subset O_{s_j} \times (Y_{V'} \cap O_{\mathbf{s}'}) = O_{s_j} \times \iota(\pi_{\cup_{s'_i = n_i} E_i}(V)) \times \prod_{s'_i < n_i} O_{s'_i} \\ &= \iota(\pi_{\cup_{s_i = n_i} E_i}(V)) \times \prod_{s_i < n_i} O_{s_i}. \end{aligned}$$

The dimension of the left-hand set is at least $\text{rk}(\mathbf{s})$ because V is polymatroid general; the right-hand set is connected of dimension $\text{rk}(\mathbf{s})$ by Lemma 3.2. Hence, the two sides are equal.

To see that $Y_V \cap O_{\mathbf{s}}$ is a single V -orbit, recall that V is polymatroid general, meaning that the V -orbit of any point in $O_{\mathbf{s}}$ has dimension $\text{rk}(\mathbf{s})$, and we have just proved that $Y_V \cap O_{\mathbf{s}}$ is connected of dimension $\text{rk}(\mathbf{s})$. The rest of the “consequently” is clear. \square

LEMMA 4.12. *If \mathbf{s} is a combinatorial flat, then $Y_V \cap O_{\mathbf{s}}$ is nonempty.*

Proof. We induct on $\dim V$. If $\dim V = 1$, let \mathbf{s} be its minimal combinatorial flat. Since V is just a line, one sees that $O_{\mathbf{s}} \subset Y_V$ using the explicit description of the Bell polynomials in Section 4.1.

Otherwise, suppose $\dim V > 1$. For a general hyperplane H in $\mathbb{K}^{\mathbf{n}}$, $Y_V \cap H \cap O_{\mathbf{s}}$ is nonempty for every combinatorial flat \mathbf{s} of corank greater than 1 by the induction hypothesis and Lemma 3.10. By the same argument, if $F = \{i : s_i = n_i\}$ is nonempty, then $Y_V \cap H \cap O_{\mathbf{s}}$ is nonempty for $H \subset \mathbb{K}^{\mathbf{n}}$ the preimage of a general hyperplane in $\mathbb{K}^{\pi_F(\mathbf{n})}$.

We have now reduced to the case where \mathbf{s} is a combinatorial flat of rank $d - 1$ with $\mathbf{s}^{\text{geo}} = \mathbf{0}$. By Lemma 3.4, this means that \mathbf{s} is an independent multiset of cardinality $d - 1$, so there is $i \in E$ such that $\mathbf{s} + \mathbf{e}_i$ is a basis of P . Let $H = \mathbb{P}^{\mathbf{n} - \mathbf{e}_i}$. Every irreducible component of $Y_V \cap H$ is $(d - 1)$ -dimensional, so by [12, Theorem 1.26],

$$[\mathbb{P}^{\mathbf{n} - \mathbf{e}_i}] \cdot [Y_V] = \sum_C m_C [C],$$

where C runs over all irreducible components of $Y_V \cap H$. By Lemma 4.10,

$$Y_V \cap H \subset \bigcup_{\mathbf{s}'} Y_V \cap \mathbb{P}^{\mathbf{s}'},$$

where the union runs over all corank 1 combinatorial flats $\mathbf{s}' \leq \mathbf{n} - \mathbf{e}_i$. By Lemma 4.11, each irreducible component C of $Y_V \cap H$ is among those of $\bigcup_{\mathbf{s}'} Y_V \cap \mathbb{P}^{\mathbf{s}'}$, so

$$[\mathbb{P}^{\mathbf{n} - \mathbf{e}_i}] \cdot [Y_V] = \sum_{\mathbf{s}'} m_{\mathbf{s}'} [Y_V \cap \mathbb{P}^{\mathbf{s}'}],$$

with $m_{\mathbf{s}'} = 0$ if $\dim Y_V \cap \mathbb{P}^{\mathbf{s}'} < d - 1$. By Lemma 4.11 and Lemma 4.6, we may further expand as

$$[\mathbb{P}^{\mathbf{n} - \mathbf{e}_i}] \cdot [Y_V] = \sum_{\mathbf{s}'} m_{\mathbf{s}'} \sum_{\mathbf{b}' \text{ basis of } \mathbf{s}'} c_{\mathbf{s}', \mathbf{b}'} [\mathbb{P}^{\mathbf{b}'}].$$

On the other hand, by Lemma 4.6,

$$[\mathbb{P}^{\mathbf{n} - \mathbf{e}_i}] \cdot [Y_V] = \sum_{\substack{\mathbf{b} \text{ basis of } P \\ b_i > 0}} c_{\mathbf{b}} [\mathbb{P}^{\mathbf{b} - \mathbf{e}_i}],$$

with all coefficients positive. Since \mathbf{s} is both an independent multiset and a combinatorial flat of P , $[\mathbb{P}^{\mathbf{s}}]$ appears in both expansions, and $0 < c_{\mathbf{s}} = m_{\mathbf{s}} c_{\mathbf{s}, \mathbf{s}}$. This implies $m_{\mathbf{s}} \neq 0$, so $\dim Y_V \cap \mathbb{P}^{\mathbf{s}} = d - 1$, hence $Y_V \cap O_{\mathbf{s}} \neq \emptyset$. \square

Proof of Theorem 1.3. Characterization of when Y_V intersects $O_{\mathbf{s}}$ is Lemma 4.10 and Lemma 4.12. Theorem 1.3(1) follows from Lemma 4.11, and Theorem 1.3(3) is obtained by combining Lemma 4.11 and Lemma 4.6.

It remains to prove Theorem 1.3(2), which states that if $\mathbf{s}, \mathbf{s}' \in \mathcal{L}_{P, \mathbf{n}}$, then $\overline{Y_V \cap O_{\mathbf{s}}} \supset Y_V \cap O_{\mathbf{s}'}$ if and only if $\mathbf{s} \geq \mathbf{s}'$.

Fix $\mathbf{s} \in \mathcal{L}_{P, \mathbf{n}}$. By Lemma 3.9, $\mathbf{s}' < \mathbf{s}$ is a combinatorial flat of V if and only if $\mathbf{s}'_i \leq \mathbf{s}_i$ for all i and $(\mathbf{s}'_j)_{j: s_j = n_j}$ is a combinatorial flat of $V' := \pi_{\{j: s_j = n_j\}}(V)$. Such multisets \mathbf{s}' are precisely those that index the nonempty strata of $\overline{Y_V \cap O_{\mathbf{s}}}$, since

$$\overline{Y_V \cap O_{\mathbf{s}}} = Y_{V'} \times \prod_{s_i < n_i} \mathbb{P}^{s_i}$$

by Lemma 4.11. Lemma 4.11 also implies

$$O_{\mathbf{s}'} \cap (Y_{V'} \times \prod_{s_i < n_i} \mathbb{P}^{s_i}) = O_{\mathbf{s}'} \cap Y_V$$

whenever both sides are nonempty, which completes the proof. \square

REMARK 4.13 (Rescaling coordinates). Theorems 1.3 describes properties of Schubert varieties of subspace arrangements, extending well-known results for Schubert varieties of hyperplane arrangements from [2], which correspond to the case $\mathbf{n} = (1, \dots, 1)$.

One further such property follows: when $\mathbf{n} = (1, \dots, 1)$, it is well-known that changing V by rescaling each coordinate of $\mathbb{K}^{\mathbf{n}} = \mathbb{K}^N$ does not change the isomorphism class of Y_V . For general \mathbf{n} , the isomorphism class of Y_V is unchanged if we perform a weighted rescaling of each factor \mathbb{K}^{n_i} , as described in Section 4.2.

REMARK 4.14 (Flags from the group action). Assume that V realizes a simple polymatroid. The subspaces $V_{ij} = V \cap \text{St}_{j\mathbf{e}_i}$ for $1 \leq i \leq N$ and $1 \leq j \leq n_i$ comprise a collection of N flags in V . The combinatorial flats record intersection pattern of these flags, as $\text{rk}(\mathbf{s}) = \text{codim}_V V_{1,s_1} \cap V_{2,s_2} \cap \dots \cap V_{N,s_N}$. The “polymatroid general” condition on V guarantees that these flags are in general position with respect to one another (see also Remark 3.7).

EXAMPLE 4.15 (Simplification changes the Schubert variety). Let $\mathbf{n} = (2, 2)$, and consider the subspace $V \subset \mathbb{K}^{\mathbf{n}} = \mathbb{K}^2 \times \mathbb{K}^2$ given by the rowspan of

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

The data $V \subset \mathbb{K}^{\mathbf{n}}$ realizes the nonsimple polymatroid of Example 3.18, whose simplification is the boolean matroid on two elements. However, we will see that Y_V is not isomorphic to the Schubert variety $(\mathbb{P}^1)^2$ of the simplification. Consider the linear parameterization $(u, v) \mapsto (u, v, u + v, u - v)$ of V , and the corresponding parameterization of $\iota(V) \subset \mathbb{P}^2 \times \mathbb{P}^2$:

$$(u, v) \mapsto [1 : u : v + \frac{1}{2}u^2] \times [1 : u + v : u - v + \frac{1}{2}(u + v)^2].$$

Recall that we label the coordinates on the right hand side as $[b_{10} : b_{11} : b_{12}] \times [b_{20} : b_{21} : b_{22}]$. The relations between the coordinates $b_{11}, b_{12}, b_{21}, b_{22}$ on the right hand side are generated by the inhomogenous equations

$$\begin{aligned} b_{21}^2 + 4b_{11} - 2b_{21} - 2b_{22}, \\ b_{11}^2 - 2b_{11} - 2b_{12} + 2b_{21}. \end{aligned}$$

To take the closure in $\mathbb{P}^{\mathbf{n}}$, we homogenize the above equations in the variables b_{10}, b_{20} so that they are bihomogeneous, and then saturate them with respect to the ideal generated by $b_{10}b_{20}$ to obtain the homogenous equations

$$\begin{aligned} 160 b_{11}b_{20}^2 - 32 b_{12}b_{20}^2 - 96 b_{10}b_{20}b_{21} + 10 b_{20}b_{21} - b_{21}^2 + 2 b_{20}b_{22}, \\ 10 b_{10}b_{11}b_{20} + b_{11}^2b_{20} - 2 b_{10}b_{12}b_{20} - 6 b_{10}^2b_{21}, \\ 240 b_{11}^3b_{20} + 48 b_{11}^2b_{12}b_{20} - 96 b_{10}b_{12}^2b_{20} - 1440 b_{10}^2b_{11}b_{21} - 288 b_{10}^2b_{12}b_{21} + 1500 b_{10}^2b_{21} \\ - 250 b_{10}b_{11}b_{21} - 25 b_{11}^2b_{21} + 50 b_{10}b_{12}b_{21} + 300 b_{10}^2b_{22}. \end{aligned}$$

One can check (e.g. by using Macaulay2 [16]) that the subvariety $Y_V \subset \mathbb{P}^{\mathbf{n}}$ defined by the displayed equations is singular, and therefore not isomorphic to $(\mathbb{P}^1)^2$.

4.8. In this section, we establish Theorem 1.7, which gives a formula for the cohomology ring of Y_V . Throughout, we assume that V is polymatroid general. For an independent multiset \mathbf{b} , let $c_{\mathbf{b}} > 0$ be the coefficient of $[\mathbb{P}^{\mathbf{b}}]$ in $[Y_V \cap \mathbb{P}^{\mathbf{b}}]$, as in the statement of Theorem 1.3.

PROPOSITION 4.16. *If $V \subset \mathbb{C}^{\mathbf{n}}$ is polymatroid general, then its singular cohomology ring $H^*(Y_V, \mathbb{Q})$ is isomorphic to $\mathbb{Q}[y_1, \dots, y_N]/I$, where I is the ideal generated by*

$$\begin{aligned} c_{\mathbf{b}'} y_1^{b'_1} \dots y_N^{b'_N} - c_{\mathbf{b}} y_1^{b_1} \dots y_N^{b_N}, \quad \mathbf{b} \text{ and } \mathbf{b}' \text{ are independent multisets of } P \text{ with } \bar{\mathbf{b}} = \bar{\mathbf{b}'}, \\ y_1^{d_1} \dots y_N^{d_N}, \quad \mathbf{d} \text{ is a dependent multiset.} \end{aligned}$$

Proof. Let $\kappa : Y_V \rightarrow \mathbb{P}^n$ denote the inclusion. By Theorem 1.3(3), the classes $\{[Y_V \cap \mathbb{P}^s] : s \in \mathcal{L}_{P,n}\}$ are linearly independent in $A_*\mathbb{P}^n$, so the pushforward $A_*Y_V \rightarrow A_*\mathbb{P}^n$ is injective.

The sets O_s and $Y_V \cap O_s$ comprise algebraic cell decompositions of \mathbb{P}^n and Y_V , respectively. Hence, the Chow groups of these varieties are naturally isomorphic to their Borel-Moore homology groups via the cycle class map [15, Example 19.1.11]. Since \mathbb{P}^n and Y_V are compact, their Borel-Moore homology is equal to their singular homology. Summing up, we've learned that the singular homology groups of Y_V are all free and that the homology pushforward $\kappa_* : H_*(Y_V; \mathbb{Z}) \rightarrow H_*(\mathbb{P}^n; \mathbb{Z})$ is injective. Applying the Universal Coefficient Theorem, we learn that $\kappa^* : H^*(\mathbb{P}^n; \mathbb{Q}) \rightarrow H^*(Y_V; \mathbb{Q})$ is surjective.

The cohomology ring of \mathbb{P}^n is isomorphic to $R := \mathbb{Q}[y_1, \dots, y_N]/(y_1^{n_1+1}, \dots, y_N^{n_N+1})$, where y_i represents a hyperplane pulled back from \mathbb{P}^{n_i} . The monomials $y_i^{n_i+1}$ are among the claimed relations because $n_i \geq \text{rk}_P(i)$ for all $1 \leq i \leq N$.

Let us verify that the remaining generators of I are in $\ker(\kappa^*)$. By the Universal Coefficient Theorem, the pullback of $\alpha \in H^k(\mathbb{P}^n; \mathbb{Q})$ is zero if and only if its cap product with any element of $H_k(Y_V; \mathbb{Q})$ is zero. By the projection formula and the injectivity of κ_* , this is equivalent to $\alpha \frown [\overline{Y_V \cap O_s}] = 0$ for all rank k combinatorial flats s of P . By Theorem 1.3(3),

$$y_1^{b_1} \cdots y_N^{b_N} \frown [\overline{Y_V \cap O_s}] = \begin{cases} c_{\mathbf{b}}, & \text{if } \mathbf{b} \text{ is independent and } \overline{\mathbf{b}} = s \\ 0, & \text{otherwise,} \end{cases}$$

so all claimed relations are in $\ker(\kappa^*)$. Moreover, the dimension of the degree k homogeneous component of R/I is plainly equal to $\dim H^k(Y_V; \mathbb{Q})$, which completes the proof. \square

Proof of Theorem 1.7. Let $S = \mathbb{Q}[y_s : s \in \mathcal{L}_{P,n}]$, graded by $\deg y_s = \text{rk}_P(s)$. Using the notation of Proposition 4.16, define a map

$$\phi : S \rightarrow H^*(Y_V, \mathbb{Q}), \quad y_s \mapsto \frac{1}{c_{\mathbf{b}}} y_1^{b_1} \cdots y_N^{b_N}$$

where \mathbf{b} is any basis of s . The image of y_s does not depend on \mathbf{b} by Proposition 4.16.

Let J be the ideal of S generated by

$$y_s y_{s'}, \quad \text{rk}(s \smile s') < \text{rk}(s) + \text{rk}(s')$$

$c_{\mathbf{b}} c_{\mathbf{b}'} y_s y_{s'} - c_{\mathbf{b}+\mathbf{b}'} y_{s \smile s'}$, \mathbf{b}, \mathbf{b}' are bases of s, s' so that $\mathbf{b} + \mathbf{b}'$ is a basis for $s \smile s'$.

We first show that $J \subset \ker \phi$. Suppose s and s' are two combinatorial flats. If $\text{rk}(s \smile s') < \text{rk}(s) + \text{rk}(s')$, then the sum of any pair of bases of s and s' is a dependent multiset, so $\phi(y_s y_{s'}) = 0$. On the other hand, if $\text{rk}(s \smile s') = \text{rk}(s) + \text{rk}(s')$, then there are bases \mathbf{b} and \mathbf{b}' such that $\mathbf{b} + \mathbf{b}'$ is a basis for $s \smile s'$, so $c_{\mathbf{b}} c_{\mathbf{b}'} \phi(y_s y_{s'}) = c_{\mathbf{b}+\mathbf{b}'} \phi(y_{s \smile s'})$. Hence, $J \subset \ker \phi$.

Surjectivity of ϕ follows from the fact that $\phi(y_s) \frown [Y_V \cap \mathbb{P}^{s'}]$ is 1 if $s = s'$ and 0 otherwise. Finally, it is evident from the relations that the y_s generate S/J as a \mathbb{Q} -vector space, so the degree k part of S/J has dimension at most $\dim H^k(Y_V; \mathbb{Q})$, and therefore $J = \ker \phi$. \square

REMARK 4.17 (Geometry & top-heaviness of combinatorial flats). In general, our proof of Theorem 1.1(4) (Section 3.4) relies on high-powered results of [5]. However, if P is realized by $V \subset \mathbb{K}^n$ with \mathbb{K} of characteristic 0, then our work gives an independent proof of top-heaviness for \mathcal{L}_P via arguments similar to those contained in [19, 3].

In outline: By [24, Proposition 6.8.11], we may assume that $\mathbb{K} = \mathbb{C}$. Let $IH^i(Y_V)$ be the degree i intersection cohomology of Y_V , and let $H^i(Y_V)$ be the degree i singular

cohomology. Arguments of [19, 3] show that there is an injection of $H^*(Y_V)$ -modules $H^*(Y_V) \rightarrow IH^*(Y_V)$. Intersection cohomology has the Hard Lefschetz property for any ample class $L \in H^2(Y_V)$, meaning that the multiplication maps

$$IH^i(Y_V) \rightarrow IH^{d-i}(Y_V), \quad \alpha \mapsto L^{d-2i}\alpha,$$

are injective when $i \leq d/2$. These maps preserve the submodule $H^*(Y_V) \subset IH^*(Y_V)$, so they restrict to injections $H^i(Y_V) \rightarrow H^{d-i}(Y_V)$. Theorem 1.7 states that $\dim H^k(Y_V) = |\mathcal{L}_P^k|$, so taking dimensions yields top-heaviness for \mathcal{L}_P .

5. PROBLEMS

5.1. We produce Schubert varieties for polymatroids P realizable over a field of characteristic 0. In this case, our work provides a geometric proof of the top-heavy property for \mathcal{L}_P . The proof of top-heaviness for realizable matroids [19] relies on being able to construct Schubert varieties of hyperplane arrangements in positive characteristic.

QUESTION 5.1. *Given a polymatroid P realizable over a field of positive characteristic, construct a variety with an affine paving whose poset of strata is the lattice of combinatorial flats, thereby proving top-heaviness of \mathcal{L}_P .*

5.2. Schubert varieties of hyperplane arrangements are always normal, but this is untrue for Schubert varieties of subspace arrangements.

EXAMPLE 5.2. Let V be cut out by the equation $a_{11} + a_{12} = a_{21} + a_{22}$ in $\mathbb{C}^2 \times \mathbb{C}^2$. Then Y_V is the hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$b_{20}^2(b_{10}b_{11} + b_{10}b_{12} - \frac{1}{2}b_{11}^2) = b_{10}^2(b_{20}b_{21} + b_{20}b_{22} - \frac{1}{2}b_{21}^2).$$

Setting $b_{ij/k\ell} := b_{ij}/b_{k\ell}$, let U be, for example, the affine chart on which $b_{12} \neq 0$ and $b_{22} \neq 0$, with coordinate ring $\mathcal{O}(U)$ given by

$$\frac{\mathbb{K}[b_{10/12}, b_{11/12}, b_{20/22}, b_{21/22}]}{\langle b_{20/22}^2(b_{10/12}b_{11/12} + b_{10/12} - \frac{1}{2}b_{11/12}^2) - b_{10/12}^2(b_{20/22}b_{21/22} + b_{20/22} - \frac{1}{2}b_{21/22}^2) \rangle}.$$

The rational function

$$f = \frac{b_{20/22}}{b_{10/12}}(b_{10/12}b_{11/12} + b_{10/12} - \frac{1}{2}b_{11/12}^2)$$

satisfies the monic polynomial

$$X^2 - (b_{10/12}b_{11/12} + b_{10/12} - \frac{1}{2}b_{11/12}^2)(b_{20/22}b_{21/22} + b_{20/22} - \frac{1}{2}b_{21/22}^2) \in \mathcal{O}(U)[X],$$

yet is not a regular function on U . Nevertheless, in general we conjecture that $Y_V \subset \prod_{i=1}^N \mathbb{P}^{n_i}$ is normal when the subspaces $V \cap \ker(\pi_{E \setminus i})$, $1 \leq i \leq N$, span V .

QUESTION 5.3. *Determine when Schubert varieties of subspace arrangements are normal.*

5.3. The ideals defining Schubert varieties of hyperplane arrangements in $(\mathbb{P}^1)^N$ are studied in [2], and found to be highly tractable: multidegrees, multigraded Betti numbers, and initial ideals can all be understood explicitly in terms of the combinatorics of the relevant matroid. We believe these objects will also admit combinatorial descriptions for Schubert varieties of subspace arrangements. A major obstacle to verifying this is that we do not know their defining equations.

QUESTION 5.4. *Find defining equations for Schubert varieties of subspace arrangements.*

Unlike those of Schubert varieties of hyperplane arrangements, equations for Schubert varieties of subspace arrangements cannot be obtained by mere homogenization of circuits, even in relatively simple cases.

EXAMPLE 5.5. Let V be the line in \mathbb{C}^3 spanned by $(1, 1, 1)$, which represents the polymatroid P of rank 1 on one element. A minimal set of polynomials defining $Y_V \subset \mathbb{P}^3$ is:

$$\begin{aligned} 6x_0b_{12} + 4x_1b_{12} + 2b_{12}^2 - 6x_0x_3 - 3x_1x_3 \\ x_1^2 - 2x_1b_{12} - 2b_{12}^2 + 3x_1x_3 \\ 3x_0x_1 + 5x_1b_{12} + 4b_{12}^2 - 3x_0x_3 - 6x_1x_3. \end{aligned}$$

The ideal generated by these polynomials is not the ideal of homogenized circuits of V . However, observe that there are three minimal equations of degree 2, and that the matroid lift \tilde{P} with respect to $\mathbf{n} = (3)$ is a uniform matroid of rank 1 on three elements, which has three circuits of size 2.

In general, we observe in examples that if $Y_V \subset \mathbb{P}^{\mathbf{n}}$, then the number of minimal equations of a given multidegree \mathbf{d} is equal to the number of circuits of the lift \tilde{P} with respect to \mathbf{n} that represent \mathbf{d} .

Along similar lines, [2, Theorem 1.5] gives a combinatorial formula for the multi-graded Betti numbers of the ideal of a Schubert variety of hyperplane arrangement.

QUESTION 5.6. *Find a combinatorial formula for the multigraded Betti numbers of the ideal of a Schubert variety of subspace arrangement.*

5.4. The Chow class of a Schubert variety of subspace arrangement $Y_V \subset \mathbb{P}^{\mathbf{n}}$ may not be multiplicity-free, and the coefficients $c_{\mathbf{b}}$ of $[Y_V]$ may depend on \mathbf{n} . We conjecture the following formula:

CONJECTURE 5.7. *If $Y_V \subset \mathbb{P}^{\mathbf{n}}$, then when \mathbf{b} is a basis, the coefficient of $[\mathbb{P}^{\mathbf{b}}]$ in $[Y_V]$ is $c_{\mathbf{b}} = \binom{n_1}{b_1} \binom{n_2}{b_2} \cdots \binom{n_N}{b_N}$.*

5.5. The singularities of Schubert varieties of hyperplane arrangements can be resolved in a “universal fashion” using the **stellahedral variety** of [6].

QUESTION 5.8. *Resolve the singularities of Schubert varieties of subspace arrangements. Is there a “universal” resolution?*

5.6. The **Z-polynomial** of a matroid M is a combinatorial invariant defined in [25]. When M is realized by $V \subset \mathbb{K}^N$, the Z-polynomial is the Poincaré polynomial of the intersection cohomology of the Schubert variety of hyperplane arrangement Y_V .

QUESTION 5.9. *Compute the intersection cohomology Betti numbers of a Schubert variety of subspace arrangement. Are they combinatorially determined?*

An analogous question can be asked for the **Kazhdan–Lusztig polynomial** of [13], which in the case of a realizable matroid corresponds to the local intersection cohomology of Y_V at the point (∞, \dots, ∞) .

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